Note on Radius of Posets Whose Double Bound Graphs Are the Same

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Abstract

In this paper, we consider some properties on a family $\mathcal{P}_{\mathrm{DB}}(G)$ of posets whose corresponding double bound graphs are the same graph G. We deal with distances of posets in $\mathcal{P}_{\mathrm{DB}}(G)$. Furthermore we consider the radius of $\mathcal{P}_{\mathrm{DB}}(G)$ and extremal graphs on radii.

Introduction

In this paper, we consider finite undirected simple graphs. For a poset $P = (X, \leq)$ and $x \in X$, $L_P(x) = \{y \in X \ ; \ y < x\}$ and $U_P(x) = \{y \in X \ ; \ y > x\}$. $\mathrm{Max}(P)$ is the set of all maximal elements of P. $\mathrm{Min}(P)$ is the set of all minimal elements of P.

For a poset $P = (X, \leq)$, the *double bound graph* (*DB-graph*) is the graph $\mathrm{DB}(P) = (X, E_{\mathrm{DB}(P)})$, where $uv \in E_{\mathrm{DB}(P)}$ if and only if $u \neq v$ and there exist $m, n \in X$ such that $n \leq u, v \leq m$. We say that a graph G is a *DB-graph* if there exists a poset whose double bound graph is isomorphic to G. This concept was introduced by McMorris and Zaslavsky [3].

A characterization of double bound graphs can be found in [1] as follows: For a graph G with two disjoint independent subsets M and N of $V(G)$ and $v \in V(G) - (M \cup N)$, define the sets $U(v) = \{u \in M \ ; \ uv \in E(G)\}, L(v) = \{u \in N \ ; \ uv \in E(G)\}$. A *clique* in the graph G is the vertex set of a maximal complete subgraph, and a family $\mathcal{C}$ of complete subgraphs *edge covers* G if and only if for each edge $uv \in E(G)$, there exists $C \in \mathcal{C}$ such that $u, v \in C$.

Theorem 1 (D. Diny [1]) *A graph G is a DB-graph if and only if there exists a family $\mathcal{C} = \{C_1, ..., C_n\}$ of complete subgraphs of G and disjoint independent subsets M and N such that*

(1) $\mathcal{C}$ edge covers G,

(2) for each C_i, there exist $m_i \in M$, $n_i \in N$ such that $\{m_i, n_i\} \subseteq C_i$ and $\{m_i, n_i\} \not\subseteq C_j$ for all $i \neq j$, and

(3) for each $v \in V(G)-(M \cup N)$, $|U(v)| \times |L(v)|$ equals the number of elements of $\mathcal{C}$ containing v.

Furthermore, a family $\mathcal{C}$ is the unique, minimal edge covering family of cliques in G. □

For a DB-graph G and an edge clique cover $\mathcal{C} = \{C_1, C_2, ..., C_n\}$ satisfying the conditions of Theorem 1, M is an *upper kernel* $\mathrm{UK_{DB}}(G)$ of G and N is a *lower kernel* $\mathrm{LK_{DB}}(G)$ of G. In the following sections, we consider a fixed labeled DB-graph G with a fixed upper kernel $\mathrm{UK_{DB}}(G)$ and a fixed lower kernel $\mathrm{LK_{DB}}(G)$.

For a DB-graph G, $\mathcal{P}_{\mathrm{DB}}(G) = \{P\,;\ \mathrm{DB}(P) = G, \mathrm{Max}(P) = \mathrm{UK_{DB}}(G), \mathrm{Min}(P) = \mathrm{LK_{DB}}(G)\}$. Then $\mathcal{P}_{\mathrm{DB}}(G)$ is a poset by set inclusions. In [2] and [5], we deal with a distance of posets in $\mathcal{P}_{\mathrm{DB}}(G)$ and we obtain the diameter of $\mathcal{P}_{\mathrm{DB}}(G)$. In this paper we consider some properties on the radius of $\mathcal{P}_{\mathrm{DB}}(G)$.

1. Preliminary

For a DB-graph G, the *double canonical poset* of G is the poset $d\_can(G) = (V(G), \leq_{d\_can(G)})$, where $x \leq_{d\_can(G)} y$ if and only if:

(1) $y \in \mathrm{UK_{DB}}(G)$ and $xy \in E(G)$, or
(2) $x \in \mathrm{LK_{DB}}(G)$ and $xy \in E(G)$, or
(3) $x = y$.

Then for a DB-graph G, $d\_can(G)$ is the minimum poset of $\mathcal{P}_{\mathrm{DB}}(G)$.

Let x and y be two distinct elements of poset P. Suppose that $x \notin \mathrm{Min}(P), y \notin \mathrm{Max}(P)$ and x is covered by y. Then, another poset $P^{--}_{x<y}$ is obtained from P by subtracting the relation $x \leq y$ from P, and we call this transformation the *d_deletion* of $x < y$ ($x < y$-d_deletion). Now let x and y be mutually incomparable elements in P such that $x \notin Min(P), y \notin Max(P), U_P(y) \subseteq U_P(x)$ and $L_P(y) \supseteq L_P(x)$. Then, a poset $P^{++}_{x<y}$ is obtained from P by adding the relation $x \leq y$ to P. We call this transformation the *d_addition* of $x < y$ ($x < y$-d_addition).

We easily obtain the following facts on these transformations. Any posets P and $P^{--}_{x<y}$ have the same DB-graph, and P and $P^{++}_{x<y}$ also have the same DB-graph. Moreover $x < y$-d_addition and $x < y$-d_deletion are inverse transformations of each other. By these facts, we obtain the following result on transformations between posets.

Theorem 2 (H. Era, K. Ogawa and M. Tsuchiya [2]) *Let G be a DB-graph and let P and Q be two posets in $\mathcal{P}_{\mathrm{DB}}(G)$.*

(1) P can be transformed into Q by a sequence of d_deletions and d_additions of order relations.
(2) Every poset in $\mathcal{P}_{\mathrm{DB}}(G)$ is obtained from d_can(G) *by d_additions only.* □

For a DB-graph G with an upper kernel $\mathrm{UK_{DB}}(G)$ and a lower kernel $\mathrm{LK_{DB}}(G)$, and $x \in V(G) - (\mathrm{UK_{DB}}(G) \cup \mathrm{LK_{DB}}(G))$, $\mathrm{Ma}(x) = \{y \in \mathrm{UK_{DB}}(G); xy \in E(G)\}$, $\mathrm{Mi}(x) = \{y \in \mathrm{LK_{DB}}(G); xy \in E(G)\}$. Furthermore we also know some properties on a maximal poset.

Theorem 3 (H. Era, K. Ogawa and M. Tsuchiya [2]) *Let G be a DB-graph with an upper kernel $\mathrm{UK_{DB}}(G)$ and a lower kernel $\mathrm{LK_{DB}}(G)$, and P be a maximal poset on $\mathcal{P}_{\mathrm{DB}}(G)$.*

(1) For all $x, y \in V(G) - (\mathrm{UK_{DB}}(G) \cup \mathrm{LK_{DB}}(G))$ such that $Ma(x) \neq Ma(y)$ or $Mi(x) \neq Mi(y)$, $x \leq_P y$ if and only if $Ma(x) \supseteq Ma(y)$ and $Mi(x) \subseteq Mi(y)$.
(2) For all $x, y \in V(G) - (\mathrm{UK_{DB}}(G) \cup \mathrm{LK_{DB}}(G))$ such that $Ma(x) = Ma(y)$ and $Mi(x) = Mi(y)$, x is comparable with y. □

Based on these results, we deal with distances of posets whose DB-graphs are the same.

In a DB-graph G with an upper kernel $\mathrm{UK_{DB}}(G)$ and a lower kernel $\mathrm{LK_{DB}}(G)$, the *distance* between posets P and Q in $\mathcal{P}_{\mathrm{DB}}(G)$, denoted by $d_{\mathrm{DB}}(P, Q)$, is the minimum number of transformations from P to Q by d_deletions and d_additions. The *diameter* $d(X)$ is $max\{d_{\mathrm{DB}}(P, Q); P, Q \in X\}$. For a DB-graph with an upper kernel $\mathrm{UK_{DB}}(G)$ and a lower kernel $\mathrm{LK_{DB}}(G)$, and two subsets $S \subseteq \mathrm{UK_{DB}}(G), T \subseteq \mathrm{LK_{DB}}(G)$,

$$I(S,T) = \bigcap_{\substack{m \in S,\\ n \in T}} N_G(n,m) - \bigcup_{\substack{m \in \mathrm{UK_{DB}}(G)-S,\\ n \in \mathrm{LK_{DB}}(G)-T}} N_G(n,m),$$

where $N_G(n,m) = \{v \in V(G); vn, vm \in E(G)\}$. For a DB-graph with an upper kernel $\mathrm{UK_{DB}}(G)$ and a lower kernel $\mathrm{LK_{DB}}(G)$, let Σ_1^d and Σ_2^d denote the following:

$$\Sigma_1^d = \sum_{\substack{\emptyset \neq S \subseteq \mathrm{UK_{DB}}(G),\\ \emptyset \neq T \subseteq \mathrm{LK_{DB}}(G)}} \binom{|I(S,T)|}{2},$$

$$\Sigma_2^d = \sum_{\substack{\emptyset \neq S_1 \subseteq S_2 \subseteq \mathrm{UK_{DB}}(G),\\ \emptyset \neq T_2 \subseteq T_1 \subseteq \mathrm{LK_{DB}}(G),\\ \text{and}\\ S_1 \neq S_2 \text{ or } T_1 \neq T_2}} (|I(S_1, T_1)| \times |I(S_2, T_2)|)$$

Then Σ_1^d is the number of pairs $x, y \in V(G) - (\mathrm{UK_{DB}}(G) \cup \mathrm{LK_{DB}}(G))$ such that $\mathrm{Ma}(x) = \mathrm{Ma}(y)$ and $\mathrm{Mi}(x) = \mathrm{Mi}(y)$, and Σ_2^d is the number of pairs $x, y \in V(G) - (\mathrm{UK_{DB}}(G) \cup \mathrm{LK_{DB}}(G))$ such that $\mathrm{Ma}(x) \neq \mathrm{Ma}(y)$ or $\mathrm{Mi}(x) \neq \mathrm{Mi}(y)$, and $\mathrm{Ma}(x) \subseteq \mathrm{Ma}(y)$ and $\mathrm{Mi}(x) \supseteq \mathrm{Mi}(y)$.

For $P, Q \in \mathcal{P}_{\mathrm{DB}}(G)$, $[P, Q] = \{R \in \mathcal{P}_{\mathrm{DB}}(G); P \leq_{\mathcal{P}_{\mathrm{DB}}(G)} R \leq_{\mathcal{P}_{\mathrm{DB}}(G)} Q\}$. We already know the followings.

Theorem 4 (H. Era, K. Ogawa and M. Tsuchiya [2]) *Let G be a DB-graph with an upper kernel $\mathrm{UK_{DB}}(G)$ and a lower kernel $\mathrm{LK_{DB}}(G)$, and $P_{\max}^d$ be a maximal poset in $\mathcal{P}_{\mathrm{DB}}(G)$. Then*

$$d([\mathrm{d\_can}(G), P_{\max}^d]) = \Sigma_1^d + \Sigma_2^d.$$

□

Theorem 5 (H. Era, K. Ogawa and M. Tsuchiya [2]) *For a DB-graph G with an upper kernel* $\mathrm{UK_{DB}}(G)$ *and a lower kernel* $\mathrm{LK_{DB}}(G)$,

$$d(\mathcal{P}_{\mathrm{DB}}(G)) = 2\Sigma_1^d + \Sigma_2^d.$$

□

2. The radius of $\mathcal{P}_{\mathrm{DB}}(G)$

In this section, we consider the radius of $\mathcal{P}_{\mathrm{DB}}(G)$. The *eccentricity* $e(P)$ is $max\{d_{\mathrm{DB}}(P,Q); Q \in \mathcal{P}_{\mathrm{DB}}(G)\}$. The *radius* $r(\mathcal{P}_{\mathrm{DB}}(G))$ is $min\{e(P); P \in \mathcal{P}_{\mathrm{DB}}(G)\}$. We have the following result.

Proposition 6 *For a DB-graph G with an upper kernel* $\mathrm{UK_{DB}}(G)$ *and a lower kernel* $\mathrm{LK_{DB}}(G)$,

$$\Sigma_1^d + \lceil \Sigma_2^d/2 \rceil \leq r(\mathcal{P}_{\mathrm{DB}}(G)) \leq \Sigma_1^d + \Sigma_2^d.$$

Proof. By the definitions of the diameter and the radius, $r(\mathcal{P}_{\mathrm{DB}}(G)) \leq d(\mathcal{P}_{\mathrm{DB}}(G)) \leq 2r(\mathcal{P}_{\mathrm{DB}}(G))$. Therefore $\Sigma_1^d + \lceil \Sigma_2^d/2 \rceil = \lceil d(\mathcal{P}_{\mathrm{DB}}(G))/2 \rceil \leq r(\mathcal{P}_{\mathrm{DB}}(G))$. By Theorem 4, $d([\mathrm{d\_can}(G), P^d_{\mathrm{max}}]) = \Sigma_1^d + \Sigma_2^d$ and $e(\mathrm{d\_can}(G)) = \Sigma_1^d + \Sigma_2^d$. Thus $r(\mathcal{P}_{\mathrm{DB}}(G)) \leq \Sigma_1^d + \Sigma_2^d$. □

Next we consider graphs for which the equalities in Proposition 6 are hold. We have the following results.

Proposition 7 *Let G be a DB-graph with an upper kernel* $\mathrm{UK_{DB}}(G)$ *and a lower kernel* $\mathrm{LK_{DB}}(G)$. *For any elements $x, y \in V(G) - (\mathrm{UK_{DB}}(G) \cup \mathrm{LK_{DB}}(G))$ such that* $\mathrm{Ma}(x) \supseteq \mathrm{Ma}(y)$ *and* $\mathrm{Mi}(x) \subseteq \mathrm{Mi}(y)$, *if there exist no elements $z \in V(G) - (\mathrm{UK_{DB}}(G) \cup \mathrm{LK_{DB}}(G))$ such that* $\mathrm{Ma}(x) \supseteq \mathrm{Ma}(z) \supseteq \mathrm{Ma}(y)$ *and* $\mathrm{Mi}(x) \subseteq \mathrm{Mi}(z) \subseteq \mathrm{Mi}(y)$, *then* $r(\mathcal{P}_{\mathrm{DB}}(G)) = \Sigma_1^d + \Sigma_2^d$.

Proof. Let G be a graph satisfying the conditions. For a comparable pair $x, y \in V(G) - (\mathrm{UK_{DB}}(G) \cup \mathrm{LK_{DB}}(G))$ of a poset $P \in \mathcal{P}_{\mathrm{DB}}(G)$ such that $\mathrm{Ma}(x) \supseteq \mathrm{Ma}(y)$ and $\mathrm{Mi}(x) \subseteq \mathrm{Mi}(y)$, we can obtain $P^{--}_{x<y}$ by a d_deletion on $x < y$. Furthermore for an incomparable pair $x, y \in V(G) - (\mathrm{UK_{DB}}(G) \cup \mathrm{LK_{DB}}(G))$ of a poset $P \in \mathcal{P}_{\mathrm{DB}}(G)$ such that $\mathrm{Ma}(x) \supseteq \mathrm{Ma}(y)$ and $\mathrm{Mi}(x) \subseteq \mathrm{Mi}(y)$, we can obtain $P^{++}_{x<y}$ by an addition on $x < y$. Therefore for a poset $P \in \mathcal{P}_{\mathrm{DB}}(G)$, we can obtain a poset $Q \in \mathcal{P}_{\mathrm{DB}}(G)$ such that if $x \leq_P y$ and $\mathrm{Ma}(x) \supseteq \mathrm{Ma}(y)$ and $\mathrm{Mi}(x) \subseteq \mathrm{Mi}(y)$, then x is incomparable to y in Q, and if x is incomparable to y in P and $\mathrm{Ma}(x) \supseteq \mathrm{Ma}(y)$ and $\mathrm{Mi}(x) \subseteq \mathrm{Mi}(y)$, then $x \leq_Q y$. Hence for any poset $P \in \mathcal{P}_{\mathrm{DB}}(G)$, $e(P) \geq \Sigma_1^d + \Sigma_2^d$. Thus $r(\mathcal{P}_{\mathrm{DB}}(G)) \geq \Sigma_1^d + \Sigma_2^d$. By Theorem 4, $r(\mathcal{P}_{\mathrm{DB}}(G)) = \Sigma_1^d + \Sigma_2^d$. □

We define a special poset in $\mathcal{P}_{\mathrm{DB}}(G)$. For a DB-graph G, the *double neutral poset* of G is the poset $\mathrm{d\_neu}(G) = (V(G), \leq_{\mathrm{d\_neu}(G)})$, where $x \leq_{\mathrm{d\_neu}(G)} y$ if and only if

(1) $y \in \mathrm{UK_{DB}}(G)$ and $xy \in E(G)$, or
(2) $x \in \mathrm{LK_{DB}}(G)$ and $xy \in E(G)$, or
(3) $x = y$, or
(4)(a) $\mathrm{Ma}(x) \neq \mathrm{Ma}(y)$ or $\mathrm{Mi}(x) \neq \mathrm{Mi}(y)$, and
(b) $\mathrm{Ma}(x) \supseteq \mathrm{Ma}(y)$, and
(c) $\mathrm{Mi}(x) \subseteq \mathrm{Mi}(y)$.

We have the following result.

Lemma 8 *For a poset P in $\mathcal{P}_{\mathrm{DB}}(G)$, there exists a poset Q in $[\mathrm{d\_can}(G), \mathrm{d\_neu}(G)]$ such that for any $x, y \in V(Q), x \leq_Q y$ if and only if $x \leq_P y$ and $x \leq_{\mathrm{d\_neu}(G)} y$.*

Proof. We construct a poset Q from P as follows: (1)$V(P) = V(Q)$ (2)if $x \leq_P y$ and $x \leq_{\mathrm{d\_neu}(G)} y$, then $x \leq_Q y$. Then Q is the intersection of P and $\mathrm{d\_neu}(G)$. Therefore Q is a poset in $\mathcal{P}_{\mathrm{DB}}(G)$ and $Q \in [\mathrm{d\_can}(G), \mathrm{d\_neu}(G)]$. □

Noting that a poset Q of Lemma 8 is obtained from P by d_deletions only, we know that the poset P is obtained from Q by d_additions only. Thus we have the following result.

Proposition 9 *Let G be a DB-graph with an upper kernel $\mathrm{UK_{DB}}(G)$ and a lower kernel $\mathrm{LK_{DB}}(G)$. If $V(G) - (\mathrm{UK_{DB}}(G) \cup \mathrm{LK_{DB}}(G)) = \{u_1, u_2, u_3, v_1, ..., v_n\}$ satisfies the following conditions, then $r(\mathcal{P}_{\mathrm{DB}}(G)) = \Sigma_1^d + \lceil \Sigma_2^d/2 \rceil$:*

(1) $\mathrm{Ma}(u_i) \supseteq \mathrm{Ma}(u_j)$ *and* $\mathrm{Mi}(u_i) \subset \mathrm{Mi}(u_j)$, *or* $\mathrm{Ma}(u_i) \supset \mathrm{Ma}(u_j)$ *and* $\mathrm{Mi}(u_i) \subseteq \mathrm{Mi}(u_j)$ *for* $1 \leq i < j \leq 3$,
(2) $\mathrm{Ma}(u_i) \parallel \mathrm{Ma}(v_j)$ *or* $\mathrm{Mi}(u_i) \parallel \mathrm{Mi}(v_j)$ *for* $^\forall i = 1, 2, 3$ *and* $^\forall j = 1, ..., n$, *or*
(3) $\mathrm{Ma}(v_i) \not\supset \mathrm{Ma}(v_j)$ *and* $\mathrm{Mi}(v_i) \not\subset \mathrm{Mi}(v_j)$ *for* $^\forall i, j (i \neq j)$.

Proof. By Lemma 8, for a poset $P \in \mathcal{P}_{\mathrm{DB}}(G)$, there exists a poset $Q \in [\mathrm{d\_can}(G), \mathrm{d\_neu}(G)]$ such that $d_{\mathrm{DB}}(P, Q) \leq \Sigma_1^d$. By the condition (1), $\Sigma_2^d = 3$. Let R be a poset in $\mathcal{P}_{\mathrm{DB}}(G)$, whose relations are the relations of $\mathrm{d\_can}(G)$ and the followings: $u_1 \leq_R u_3$, $u_1 \not\leq_R u_2$ and $u_2 \not\leq_R u_3$. Then for any poset $Q \in [\mathrm{d\_can}(G), \mathrm{d\_neu}(G)]$, $d_{\mathrm{DB}}(Q, R) \leq 2 = \lceil \Sigma_2^d/2 \rceil$. Therefore for a poset $P \in \mathcal{P}_{\mathrm{DB}}(G)$, $d_{\mathrm{DB}}(P, R) \leq d_{\mathrm{DB}}(P, Q) + d_{\mathrm{DB}}(Q, R) \leq \Sigma_1^d + \lceil \Sigma_2^d/2 \rceil$. By Proposition 6, $r(\mathcal{P}_{\mathrm{DB}}(G)) = \Sigma_1^d + \lceil \Sigma_2^d/2 \rceil$. □

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(論文：同じ双界グラフを持つ半順序集合の族の半径について　著者：いわい しんいち，おがわ けんじろう，東海大学大学院理学研究科，つちやもりまさ，東海大学理学部　受付：平成12年12月22日)