Complete Bipartite Geometric Graphs and Alternating Paths

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Abstract

Let A and B be two disjoint sets of points in the plane such that no three points of $A \cup B$ are collinear, and let n be the number of points in A. A geometric complete bipartite graph $K(A, B)$ is a complete bipartite graph with partite sets A and B which is drawn in the plane such that each edge of $K(A, B)$ is a straight-line segment. We prove that

(i) If $|B| \geq (n+1)(2n-4)+1$, then the geometric complete bipartite graph $K(A, B)$ contains a path P without crossings such that $V(P)$ contains the set A.

(ii) There exists a configuration of $A \cup B$ with $|B| = \frac{n^2}{16} + \frac{n}{2} - 1$ such that in $K(A, B)$ every path containing the set A has at least one crossing.

1 Introduction

Let G be a finite graph without loops or multiple edges. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of G respectively. For a vertex v of G we denote by $\deg_G(v)$ the degree of v in G. For a set X we denote by $|X|$ the cardinality of X. *A geometric graph* $G=(V(G), E(G))$ is a graph drawn in the plane such that $V(G)$ is a set of points in the plane, no three of which are collinear, and $E(G)$ is a set of (possibly crossing) straight-line segments whose endpoints belong to $V(G)$. If a geometric graph G is a complete bipartite graph with partite sets A and B i.e., $V(G)=A \cup B$ then G is denoted by $K(A, B)$, which may be called a *geometric complete bipartite graph.*

In 1996, M. Abellanas, J. García, G. Hernández, M. Noy and P. Ramos [1] showed the following result.

Theorem A (Abellanas et al. [1]) *Let A and B be two disjoint sets of points in the plane such that* $|A| = |B|$ *and no three points of* $A \cup B$ *are collinear. Then the geometric complete bipartite graph K(A, B) contains a spanning tree T without crossings such that the maximum degree of T is* $O\ \log(|A|)$.

In 1999, Kaneko [3] improved their result and proved the following theorem.

Theorem B (Kaneko [3]) *Let A and B be two disjoint sets of points in the plane such that* $|A| = |B|$ *and no three points of* $A \cup B$ *are collinear. Then the geometric complete bipartite graph K(A, B) contains a spanning tree T without crossings such that the maximum degree of T is at most 3.*

It is well-known that under the same condition in Theorem *B*, there are configurations of $A \cup B$ such that $K(A, B)$ does not contain a hamiltonian path without crossings (i.e., a spanning tree of maximum degree at most 2 without crossings) (see [2]). So we are led to the following problem. Given two disjoint sets *A* and *B* of points in the plane such that no three points of $A \cup B$ are collinear, if $|B|$ is large compared with $|A|$, then does $K(A, B)$ contain a path *P* without crossings such that $V(P)$ contains the set *A*? The answer to the above question is in the affirmative, as we shall see now. We prove the following theorem.

Theorem 1 *Let A and B be two disjoint sets of points in the plane such that no three points of* $A \cup B$ *are collinear, and let n be the number of points in A.*
(i) If $|B| \geq (n+1)(2n-4)+1$, *then the geometric complete bipartite graph K(A, B) contains a path P without crossings such that V (P contains the set A.*
(ii) There exists a configuration of $A \cup B$ *with* $|B| = \frac{n^2}{16} + \frac{n}{2} - 1$ *such that in K(A, B) every path containing the set A has at least one crossing.*

In order to prove Theorem 1, we need some notation and definitions. For a set *X* of points in the plane, we denote by conv(*X*) the convex hull of *X*. The boundary of conv(*X*) is a polygon whose segments and extremes are called the *edges and the vertices* of conv(*X*), respectively. For two points *x* and *y* in the plane, we denote by *xy* the straight line segment joining *x* to *y* which may be an edge of a geometric graph containing both *x* and *y* as it vertices. Let *A* be a set of point in the plane, let *y* be a vertex of conv(*A*) and let *x* be a point exterior to conv(*A*). Then we say that *x sees y* on conv(*A*) if the line segment *xy* intersects conv(*A*) only at *y*.

Lemma 2 *Let R and S be disjoint sets of points in the plane with* $|R| \geq |S|$ *such that no three points of* $R \cup S$ *are collinear, and let l be a line in the plane separating the set R and the set S. Let x and y be two vertices of conv(*$R \cup S$*) with* $x \in R$ *and* $y \in S$ *such that xy is an edge of conv(*$R \cup S$*). Then in K(R, S) there exists a path P without crossings such that*
(i) the vertex x is an end of P, and
(ii) V(P) contains S.

Proof We prove the lemma by induction on $|R \cup S|$. If $|S| = 1$ then the lemma follows immediately, and so we may assume $|R| \geq |S| \geq 2$. Let z_0 be the vertex of conv($R \cup S$) with $z_0 \in R$ such that xz_0 is an edge of conv($R \cup S$). We consider conv($R \cup S - \{x\}$). Let $Z = \{z_1, z_2, \dots, z_m\}$ be the set of new vertices of conv($R \cup S - \{x\}$) (possibly $Z=\emptyset$), i.e., $z_1, z_2, \ldots, z_m$ are interior point of conv($R \cup S$). Set $z_{m+1} = y$. Since $|R-\{x\}| \geq 1$ and $|S|$ (≥ 2)>1, there exist two vertices of conv($R \cup S - \{x\}$) z_i and z_{i+1} such that $z_i \in R$ and $z_{i+1} \in S$, i.e., $z_i z_{i+1}$ is an edge of conv($R \cup S - \{x\}$). It is clear that the vertex x *sees* z_{i+1} on conv($R \cup S - \{x\}$). Now we consider conv($R \cup S - \{x, z_{i+1}\}$). Let $Z' = \{w_1, w_2, \ldots, w\}$ be the set of new vertices of conv($R \cup S - \{x, z_{i+1}\}$)(possibly $Z'=\emptyset$), i.e., $w_1, w_2, \ldots, w_k$ are interior points of conv($R \cup S - \{x\}$). Let w_{k+1} be the vertex of conv($R \cup S - \{x\}$) with $w_{k+1} \in S$ such that $z_{i+1} w_{k+1}$ is an edge of conv($R \cup S - \{x\}$). Set $w_0 = z_i$. Since $|R-\{x\}| \geq 1$ and $|S-\{z_{i+1}\}| \geq 1$, by repeating the above method, there exist two vertices w_j and w_{j+1} of conv($R \cup S - \{x, z_{i+1}\}$), such that $w_j \in R$ and $w_{j+1} \in S$ i.e., $w_j w_{j+1}$ is an edge of conv($R \cup S - \{x, z_{i+1}\}$), and such that the vertex z_{i+1} *sees* w_j on conv($R \cup S - \{x, z_{i+1}\}$). By the induction hypothesis, in $K(R-\{x\}, S-\{z_{i+1}\})$ there exists a path P' without crossings such that

(i) the vertex w_j ($\in R$) is an end of P', and

(ii) $V(P')$ contains $S-\{z_{i+1}\}$.

Obviously $P = P' \cup x z_{i+1} \cup z_{i+1} w_j$ is the desired path. □

Now we proceed to prove part (i) of Theorem 1. We may assume that no two points of $A \cup B$ have the same x coordinate. Let $a_1, a_2, \ldots a_n$ be points of S sorted by their x-coordinate and let P_i be the vertical line which passes through the point a_i, $1 \leq i \leq n$. These n lines separate the plane into $n+1$ regions and hence they separate the set B into $n+1$ disjoint subsets. Assume that these lines are directed upward. By the assumption, at least one subset contains at least $2n-3$ points of B. We may assume that one of the region which contains at least $2n-3$ points of B is bounded by the lines P_j and P_{j+1}, $1 \leq j \leq n-1$. (The leftmost and rightmost unbounded regions can be treated similarly.) Let B_j be the subset of B between P_j and P_{j+1} i.e., $|B_j| \geq 2n-3$. Let 10 be the line between P_j and P_{j+1} satisfying the following conditions:

(i) l_0 passes through a point b_0 of B_j and is directed upward,

(ii) Let B_l be the subset of $B_j - \{b_0\}$ to the left of l_0 and let B_r be the subset of $B_j - \{b_0\}$ to the right of l_0. Then $|B_l| \geq 2j-2$ and $|B_r| \geq 2n-2j-2$.

Let A_l be the subset of A to the left of l_0 and let A_r be the subset of A to the right of l_0. Trivially $|A_l| = j$ and $|A_r| = n-j$. Let t_1 and t_2 be the two rays emanating from b_0 such that t_i is tangent to conv(A_l) at w_i, $1 \leq i \leq 2$, and t_1 is above t_2. Also let t_3 and t_4 be the two rays emanating from b_0 such that t_i is tangent to conv(A_r) at w_i, $3 \leq i \leq 4$, and t_3 is above t_4. (Notice that since no three points of $A \cup B$ are collinear, each ray contains no point of $B_l \cup B_r$.) Let B_l^+ be the subset of B_l above the ray t_2 and B_l^- the subset of B_l under the ray t_1. Also let B_r^+ be the subset of B_r above the ray t_4 and B_r^- the subset of B_r under the ray t_3. Since $|B_l| \geq 2j-2$, we have either $|B_l^+| \geq j-1$ or $|B_l| \geq j-1$, say $|B_l^+| \geq j-1$. Similarly we have either $|B_r^+| \geq n-j-1$ or $|B_r^-| \geq n-j-1$, say $|B_r^+| \geq n-j-1$. Consider now $K(B_l^+ \cup \{b_0\}, A_l)$. Since $|B_l^+ \cup \{b_0\}| \geq j = |A_l|$, applying Lemma 2 and letting $x = b_0$, in $K(B_l^+ \cup \{b_0\}, A_l)$ we can find a path R_l without crossings such that

(i) the vertex b_0 is an end of R_l, and

(ii) $V(R_l)$ contains A_l.

In a similar manner, in $K(B_r^+) \cup \{b_0\}, A_r)$ we can find a path R_r without crossings such that

(i) the vertex b_0 is an end of R_r, and

(ii) $V(R_r)$ contains A_r.

Set $P = R_l \cup R_r$. Clearly P is a path in $K(A, B)$ without crossings such that $V(P)$ contains the set A.

In order to show part (ii) of Theorem 1, suppose that $n=4k$ and all points of $A \cup B$ lie on a cycle in the following order:

$a_1^0, a_2^0, \ldots, a_{k+2}^0, b_1^0, b_2^0, \ldots, b_k^0, a_1^1, a_2^1, b_1^1, b_2^1, \ldots, b_k^1,$
$a_1^2, a_2^2, b_1^2, b_2^2, \ldots, b_k^2, \ldots \ldots \ldots, a_1^{k-2}, a_2^{k-2}, b_1^{k-2}, b_2^{k-2}, \ldots, b_k^{k-2},$
$a_1^{k-1}, a_2^{k-1}, \ldots, a_{k+2}^{k-1}, b_1^{k-1}, b_2^{k-1}, \ldots, b_{3k-1}^{k-1},$

where a_j^i's are points in A and b_j^i's are points in B. It is not difficult to show that $|A| = n$ and $|B| = \frac{n^2}{16} + \frac{n}{2} - 1$ and that in $K(A, B)$ every path containing the set A has at least one crossing.

This completes the proof of Theorem 1. □

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References

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