A complexity analysis of a smoothing method for $P^*(\kappa)$-LCP

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Abstract

We extend the smoothing algorithm proposed in [HIY00] for monotone LCP to the one for $P^*(\kappa)$-LCP. The results of this paper are direct extension of those in [HIY00]. We show that the algorithm terminates in $O(\frac{\kappa^8\bar{\gamma}^6 n}{\epsilon^6}\log\frac{\bar{\gamma}\sqrt{n}}{\epsilon})$ Newton iterations where $\bar{\gamma}$ is a number which depends on the problem and the initial point and κ is a constant which depends on the problem.

1 Introduction

We consider the standard linear complementarity problem, LCP(q, M):

$$\begin{aligned} \text{Find} \quad & (\boldsymbol{x}, \boldsymbol{y}) \in I\!R^{2n} \\ \text{s.t.} \quad & \boldsymbol{y} = M\boldsymbol{x} + \boldsymbol{q} && (1.1) \\ & x_i y_i = \mu \ (i \in N) && (1.2) \\ & (\boldsymbol{x}, \boldsymbol{y}) \geq (\boldsymbol{0}, \boldsymbol{0}) && (1.3) \end{aligned}$$

where $N = \{1, 2, \ldots, n\}$, M is an $n \times n$ matrix and q is an n-dimensional vector. We assume that the following condition holds.

Condition 1.1.

(i) *Let M be an $n \times n$ matrix of the class of $P^*(\kappa)$ for some $\kappa \geq 0$, i.e., M satisfies*

$$\exists\kappa \geq 0, \forall \boldsymbol{\xi} \in I\!R^n, \ (1+4\kappa)\sum_{i \in I_+(\xi)} \xi_i [M\boldsymbol{\xi}]_i + \sum_{i \in I_-(\xi)} \xi_i [M\boldsymbol{\xi}]_i \geq 0 \quad (1.4)$$

$$\textit{where} \quad I_+(\boldsymbol{\xi}) = \{i \in N \mid \xi_i [M\boldsymbol{\xi}]_i > 0\}, \quad (1.5)$$

$$I_-(\boldsymbol{\xi}) = \{i \in N \mid \xi_i [M\boldsymbol{\xi}]_i < 0\} \quad (1.6)$$

for some $\kappa \geq 0$.

(ii) *LCP has a feasible interior point, i.e.,*

$$\exists (\mathring{\boldsymbol{x}}, \mathring{\boldsymbol{y}}) > (\boldsymbol{0}, \boldsymbol{0}) \ s.t. \ \mathring{\boldsymbol{y}} = M\mathring{\boldsymbol{x}} + \boldsymbol{q}.$$

Condition 1.1 is milder than **Condition 1** in [HIY00]. The relations among the class of $P^*(\kappa)$ matrices and other classes of matrices had been discussed (see [KMNY91]). The algorithm is based on the use of Chen-Harker-Kanzow-Smale smoothing function

$$\phi(\mu, a, b) := a + b - \sqrt{(a-b)^2 + 4\mu^2} \quad (1.7)$$

with a positive number $\mu > 0$. For $\mu > 0$, the following property holds:

$$\phi(\mu, a, b) = 0 \rightleftharpoons ab = \mu^2, \ (a, b) > (0, 0). \quad (1.8)$$

It is well-known that the next proposition for the CHKS-function.

Proposition 1.2. (**Proposition 1** of *[HIY00]*) *For every nonnegative number $\mu \geq 0$, the following equivalence* *results hold for every a, b,* $c \in I\!R$.

(i)

$$\phi(\mu, a, b) = c \rightleftharpoons (a - c/2)(b - c/2) = \mu^2 \geq 0 \text{ and } (a - c/2, b - c/2) \geq (0, 0). \tag{1.9}$$

(ii)

$$\phi(\mu, a, b) \leq 0 \rightleftharpoons ab \leq \mu^2 \text{ if } a + b \geq 0. \tag{1.10}$$

Specially, if $\mu > 0$ *then*

$$\phi(\mu, a, b) < 0 \rightleftharpoons ab < \mu^2 \text{ if } a + b > 0. \tag{1.11}$$

(iii) ϕ *is a concave function, and*

$$\nabla^2 \phi(\mu, a, b) = -\frac{4}{(\sqrt{(a-b)^2 + 4\mu^2})^3} \begin{pmatrix} a-b \\ -\mu \\ \mu \end{pmatrix} (a-b, -\mu, \mu), \tag{1.12}$$

$$\|\nabla^2 \phi(\mu, a, b)\| \leq \frac{2}{\mu}. \tag{1.13}$$

We define some symbols used throughout this paper. N means the index set $\{1, 2, \ldots, n\}$. $I\!R^n_+$ and $I\!R^n_{++}$ denote the n-dimensional nonnegative orthant and the n-dimensional positive orthant, respectively. For a given vector $\boldsymbol{x}$, vec$\{x_i\}$ and diag$\{x_i\}$ represent the n-dimensional vector whose i-th element is the i-th component of $\boldsymbol{x}$ and $n \times n$-diagonal matrix whose i-th diagonal element is the i-th component of $\boldsymbol{x}$, respectively. $\boldsymbol{e}$ denotes the vector vec$\{1\}$ whose components equal to one.

2 A smoothing algorithm for $P^*(\kappa)$-LCP

We employ the following system of equations on $(\mu, \boldsymbol{x}, \boldsymbol{y}) \in I\!R_+ \times I\!R^{2n}$ to approximate the solution of the LCP at a point $(\overline{\mu}, \overline{\boldsymbol{x}}, \overline{\boldsymbol{y}}) \in I\!R_{++} \times I\!R^{2n}$:

$$\begin{cases} \mu = \sigma_\mu \overline{\mu}, \\ \boldsymbol{y} - M\boldsymbol{x} - \boldsymbol{q} = \boldsymbol{0}, \\ \phi(\mu, x_i, y_i) = \sigma_\phi \overline{\phi}_i \quad (i \in N). \end{cases} \tag{2.1}$$

The Newton direction $(\Delta\mu, \Delta\boldsymbol{x}, \Delta\boldsymbol{y}) \in I\!R^{1+2n}$ to the system (2.1) should satisfy the following equations:

$$\Delta\mu = -(1 - \sigma_\mu)\overline{\mu} \tag{2.2}$$

$$\begin{pmatrix} -M & I \\ D_x & D_y \end{pmatrix} \begin{pmatrix} \Delta\boldsymbol{x} \\ \Delta\boldsymbol{y} \end{pmatrix} = \begin{pmatrix} \boldsymbol{0} \\ -d_\mu \Delta\mu - (1 - \sigma_\phi)\overline{\boldsymbol{\Phi}} \end{pmatrix} \tag{2.3}$$

where

$$\begin{cases} d_\mu := \text{vec}\left\{ -\frac{4}{\sqrt{(\overline{x}_i - \overline{y}_i)^2 + 4\overline{\mu}^2}} \right\}, \\ D_x := \text{diag}\left\{ 1 - \frac{\overline{x}_i - \overline{y}_i}{\sqrt{(\overline{x}_i - \overline{y}_i)^2 + 4\overline{\mu}^2}} \right\}, \\ D_y := \text{diag}\left\{ 1 + \frac{\overline{x}_i - \overline{y}_i}{\sqrt{(\overline{x}_i - \overline{y}_i)^2 + 4\overline{\mu}^2}} \right\}, \end{cases} \tag{2.4}$$

The following results are satisfied.

Proposition 2.1.

（i） *For every* $(\overline{\mu}, \overline{\boldsymbol{x}}, \overline{\boldsymbol{y}}) \in I\!R_{++} \times I\!R^{2n}$,

$$0 < 1 \pm \frac{\overline{x}_i - \overline{y}_i}{\sqrt{(\overline{x}_i - \overline{y}_i)^2 + 4\overline{\mu}_2}} < 2. \tag{2.5}$$

（ii）(*Lemma 4.1 of* [*KMNY91*]) *The system* (2.2) *and* (2.3) *has a unique solution* $(\Delta\mu, \Delta\boldsymbol{x}, \Delta\boldsymbol{y})$ *whenever Condition 1.1 holds.*

Here, we describe our algorithm in detail.

Algorithm.

Step 0: Let $\epsilon > 0$ and $k := 0$. Let $\boldsymbol{y}^0 := M\boldsymbol{x}^0 + \boldsymbol{q}$ and choose $\mu^0 > 0$ such that $(\mu^0)^2 > \max\{0, x_i^0 y_i^0\ (i \in N)\}$ and let $\phi_i^0 := \phi(\mu^0, x_i^0, y_i^0)\ (i \in N)$.

Step 1: If $\mu^k < \epsilon$ then stop else let $p := 0$ and $(\hat{\boldsymbol{x}}^0, \hat{\boldsymbol{y}}^0, \hat{\boldsymbol{\Phi}}^0) := (\boldsymbol{x}^k, \boldsymbol{y}^k, \boldsymbol{\Phi}^k)$.

step 1.1 If $\|\hat{\boldsymbol{\Phi}}^p\| < \epsilon$ then go to step 1.2. Compute the Newton direction $(\Delta\mu^p, \Delta\boldsymbol{x}^p, \Delta\boldsymbol{y}^p)$ by solving the system (2.2) and (2.3) with $\sigma_\mu := 1$ and $\sigma_\phi := 0$. Let

$$(\hat{\boldsymbol{x}}^{p+1}, \hat{\boldsymbol{y}}^{p+1}) := (\hat{\boldsymbol{x}}^p, \hat{\boldsymbol{y}}^p) + \theta^p(\Delta\boldsymbol{x}^p, \Delta\boldsymbol{y}^p),\ \hat{\boldsymbol{\Phi}}^{p+q} := \boldsymbol{\Phi}(\mu^k, \hat{\boldsymbol{x}}^{p+1}, \hat{\boldsymbol{y}}^{p+1})$$

where $\theta^p := \min\left\{1, \dfrac{\|\hat{\boldsymbol{\Phi}}^p\|\mu^k}{2(\|\Delta\boldsymbol{x}^p\|^2 + \|\Delta\boldsymbol{y}^p\|^2)}\right\}$. Let $p := p+1$ and go to step 1.1.

step1.2 Let $(\overline{\mu}, \overline{\boldsymbol{x}}, \overline{\boldsymbol{y}}) := (\mu^k, \hat{\boldsymbol{x}}^p, \hat{\boldsymbol{y}}^p)$. Compute the Newton direction $(\Delta\overline{\mu}, \Delta\overline{\boldsymbol{x}}, \Delta\overline{\boldsymbol{y}})$ by solving the system (2.2) and (2.3) with $\sigma_\mu := 1/2$ and $\sigma_\phi := 1$. Let

$$(\mu^{k+1}, \boldsymbol{x}^{k+1}, \boldsymbol{y}^{k+1}) := (\mu^k, \overline{\boldsymbol{x}}, \overline{\boldsymbol{y}}) + (\Delta\overline{\mu}, \Delta\overline{\boldsymbol{x}}, \Delta\overline{\boldsymbol{y}}).$$

Let $k := k+1$ and go to Step1.

In Condition 1.1, the $n \times n$ matrix M is supposed to be $P^*(\kappa)$ for some $\kappa \geq 0$, but this algorithm can be applied to P^0 cases. It should be noted that the value of $\kappa\ (\geq 0)$ isn't necessary on ahead in the algorithm. Since the value μ^k is always reduced to $\mu^k/2$ at each iteration k, the number of Newton iterations required is $\lceil \log \frac{\mu^0}{\epsilon} \rceil$.

3 Some basic results

In this section, we collect some basic properties of the CHKS-function ϕ and the Newton directions (2.1) under Condition 1.1.

Proposition 3.1. *Suppose that Condition 1.1 holds. Let*

$$\Lambda(\beta, \mu^0) \equiv \{(\boldsymbol{x}, \boldsymbol{y}) \in I\!R^{2n} \mid \boldsymbol{y} = M\boldsymbol{x} + \boldsymbol{q}, \boldsymbol{\Phi}(\mu, \boldsymbol{x}, \boldsymbol{y}) \leq 0, \|\boldsymbol{\Phi}(\mu, \boldsymbol{x}, \boldsymbol{y})\| \leq \beta, \mu \in (0, \mu^0]\}$$

for $\beta > 0$, $\mu^0 > 0$. *Then for* $(\boldsymbol{x}, \boldsymbol{y}) \in \Lambda(\beta, \mu^0)$,

$$-\frac{\beta}{2}e \ \leq\ \boldsymbol{x} \ \leq\ (1+4\kappa)\gamma(\beta, \mu^0)\boldsymbol{e}, \tag{3.1}$$

$$-\frac{\beta}{2}\boldsymbol{e} \ \leq\ \boldsymbol{y} \ \leq\ (1+4\kappa)\gamma(\beta, \mu^0)\boldsymbol{e} \tag{3.2}$$

where

$$\gamma(\beta, \mu^0) := \frac{n(\mu^0)^2 + \left(\mathring{\boldsymbol{x}} + \frac{\beta}{2}\boldsymbol{e}\right)^T \left(\mathring{\boldsymbol{y}} + \frac{\beta}{2}\boldsymbol{e}\right)}{\min\{\mathring{x}_i, \mathring{y}_i\}} > \frac{\beta}{2} \tag{3.3}$$

for some $\kappa > 0$.

Proof: The lower bounds of $\boldsymbol{x}$ and $\boldsymbol{y}$ directly come from (i) of Proposition 1.2 and $\phi(\mu, x_i, y_i) \geq -\beta$ for every $i \in N$. To show the upper bounds, we use the fact that Condition 1.1 holds. Since $\boldsymbol{y} = M\boldsymbol{x} + \boldsymbol{q}$, $\mathring{\boldsymbol{y}} = M\mathring{\boldsymbol{x}} + \boldsymbol{q}$ with $(\mathring{\boldsymbol{x}}, \mathring{\boldsymbol{y}}) > (\boldsymbol{0}, \boldsymbol{0})$ and M is P^*, we have

$$(1+4\kappa)\sum_{i \in I_+}(x_i - \mathring{x}_i)(y_i - \mathring{y}_i) + \sum_{i \in I_-}(x_i - \mathring{x}_i)(y_i - \mathring{y}_i) \geq 0$$

for some $\kappa \geq 0$. Thus the inequality

$$(1+4\kappa)\sum_{i \in I_+}\left\{(x_i - \frac{\phi_i}{2})(y_i - \frac{\phi_i}{2}) + (\mathring{x}_i - \frac{\phi_i}{2})(\mathring{y}_i - \frac{\phi_i}{2}) - (\mathring{x}_i - \frac{\phi_i}{2})(y_i - \frac{\phi_i}{2}) - (x_i - \frac{\phi_i}{2})(\mathring{y}_i - \frac{\phi_i}{2})\right\}$$
$$+ \sum_{i \in I_-}\left\{(x_i - \frac{\phi_i}{2})(y_i - \frac{\phi_i}{2}) + (\mathring{x}_i - \frac{\phi_i}{2})(\mathring{y}_i - \frac{\phi_i}{2}) - (\mathring{x}_i - \frac{\phi_i}{2})(y_i - \frac{\phi_i}{2}) - (x_i - \frac{\phi_i}{2})(\mathring{y}_i - \frac{\phi_i}{2})\right\} \geq 0 \tag{3.4}$$

holds for some $\kappa \geq 0$. Here, from (i) of Proposition 1.2,

$$(x_i - \phi_i/2)(y_i - \phi_i/2) = \mu^2 \leq (\mu^0)^2.$$

Since $-\beta \leq \phi_i \leq 0$ and $(\mathring{x}_i, \mathring{y}_i) > (0, 0)$,

$$(\mathring{x}_i - \phi_i/2)(\mathring{y}_i - \phi_i/2) \leq (\mathring{x}_i + \beta/2)(\mathring{y}_i + \beta/2).$$

Because $\phi_i \geq 0$,

$$\mathring{x}_i - \phi_i/2 \geq \mathring{x}_i > 0, \ \mathring{y}_i - \phi_i/2 \geq \mathring{y}_i > 0,$$

and

$$(\mathring{x}_i - \phi_i/2)(y_i - \phi_i/2) \geq \mathring{x}_i(y_i - \phi_i/2) > 0,$$
$$(x_i - \phi_i/2)(\mathring{y}_i - \phi_i/2) \geq \mathring{y}_i(x_i - \phi_i/2) > 0$$

for every $i \in N$. Applying the above some bounds to the inequality (3.4), we have

$$\begin{aligned}
0 &\leq (1+4\kappa)\sum_{i \in I_+}\left\{(\mu^0)^2 + (\mathring{x}_i + \beta/2)(\mathring{y}_i + \beta/2) - \mathring{y}_i(x_i - \phi_i/2) - \mathring{x}_i(y_i - \phi_i/2)\right\} \\
&\quad + \sum_{i \in I_-}\left\{(\mu^0)^2 + (\mathring{x}_i + \beta/2)(\mathring{y}_i + \beta/2) - \mathring{y}_i(x_i - \phi_i/2) - \mathring{x}_i(y_i - \phi_i/2)\right\} \\
&= n(\mu^0)^2 + (\mathring{\boldsymbol{x}} + \beta/2\boldsymbol{e})^T(\mathring{\boldsymbol{y}} + \beta/2\boldsymbol{e}) - \mathring{\boldsymbol{y}}^T(\boldsymbol{x} - \boldsymbol{\Phi}/2) - \mathring{\boldsymbol{x}}^T(\boldsymbol{y} - \boldsymbol{\Phi}/2) \\
&\quad + 4\kappa\sum_{i \in I_+}\left\{(\mu^0)^2 + (\mathring{x}_i + \beta/2)(\mathring{y}_i + \beta/2) - \mathring{y}_i(x_i - \phi_i/2) - \mathring{x}_i(y_i - \phi_i/2)\right\} \\
&\leqq n(\mu^0)^2 + (\mathring{\boldsymbol{x}} + \beta/2\boldsymbol{e})^T(\mathring{\boldsymbol{y}} + \beta/2\boldsymbol{e}) - \mathring{\boldsymbol{y}}^T(\boldsymbol{x} - \boldsymbol{\Phi}/2) - \mathring{\boldsymbol{x}}^T(\boldsymbol{y} - \boldsymbol{\Phi}/2) \\
&\quad + 4\kappa\sum_{i \in I_+}\left\{(\mu^0)^2 + (\mathring{x}_i + \beta/2)(\mathring{y}_i + \beta/2)\right\} \\
&\leq (1+4\kappa)\left\{n(\mu^0)^2 + (\mathring{\boldsymbol{x}} + \beta/2\boldsymbol{e})^T(\mathring{\boldsymbol{y}} + \beta/2\boldsymbol{e})\right\} - \mathring{\boldsymbol{y}}^T(\boldsymbol{x} - \boldsymbol{\Phi}/2) - \mathring{\boldsymbol{x}}^T(\boldsymbol{y} - \boldsymbol{\Phi}/2)
\end{aligned}$$

Therefore, we see that

$$x_i - \phi_i/2 \leq (1+4\kappa)\frac{1}{\mathring{y}_i}\left\{n(\mu^0)^2 + (\mathring{\boldsymbol{x}} + \beta/2\boldsymbol{e})^T(\mathring{\boldsymbol{y}} + \beta/2\boldsymbol{e})\right\} \leq (1+4\kappa)\gamma(\beta, \mu^0),$$
$$y_i - \phi_i/2 \leq (1+4\kappa)\frac{1}{\mathring{x}_i}\left\{n(\mu^0)^2 + (\mathring{\boldsymbol{x}} + \beta/2\boldsymbol{e})^T(\mathring{\boldsymbol{y}} + \beta/2\boldsymbol{e})\right\} \leq (1+4\kappa)\gamma(\beta, \mu^0).$$

∎

Proposition 3.2. *Let $(\bar{\mu}, \bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in \mathbb{R}_{++} \times \mathbb{R}^{2n}$ with $\bar{\boldsymbol{y}} = M\bar{\boldsymbol{x}} + \boldsymbol{q}$, and let $\bar{\phi}_i := \phi(\bar{\mu}, \bar{x}_i, \bar{y}_i)$ $(i \in N)$. Define $(\boldsymbol{x}', \boldsymbol{y}') := (\bar{\boldsymbol{x}} - \bar{\boldsymbol{\Phi}}/2, \boldsymbol{y} - \bar{\boldsymbol{\Phi}}/2)$. Then the following results hold.*

(i) ((*i*) *of Proposition 4 of* [*HIY00*])

$$\sqrt{(\overline{x}_i-\overline{y}_i)^2+4\mu^2}=x'_i+y'_i \ \geq\ 2\overline{\mu} \ \ (i\in N)$$

(ii) ((*ii*) *of Proposition 4 of* [*HIY00*]) *The solution of* (2.3) *is the unique solution of the system*

$$\begin{pmatrix} -M & I \\ Y' & X' \end{pmatrix}\begin{pmatrix} \Delta \boldsymbol{x} \\ \Delta \boldsymbol{y} \end{pmatrix}=\begin{pmatrix} \mathbf{0} \\ \overline{\boldsymbol{h}} \end{pmatrix} \tag{3.5}$$

where

$$X':=diag\{\boldsymbol{x}'\},\ Y':=diag\{\boldsymbol{y}'\},\ \overline{\boldsymbol{h}}:=-2(1-\sigma_\mu)\overline{\mu}^2\boldsymbol{e}-\frac{1-\sigma_\phi}{2}(X'+Y')\overline{\boldsymbol{\Phi}}.$$

(iii) *Suppose that Condition 1.1 holds. If* $(\overline{\mu}, \overline{\boldsymbol{x}}, \overline{\boldsymbol{y}})\in I\!R_{++}\times I\!R^{2n}$ *lies in the set* $\Lambda(\beta, \mu^0)$ *for some* $\beta>0$ *and* $\mu^0>0$ *then*

$$0\ <\ \frac{\overline{\mu}^2}{3(1+2\kappa)\overline{\gamma}(\beta, \mu^0)}\ \leq\ x'_i,\ y'_i\ \leq\ 2(1+2\kappa)\gamma(\beta, \mu^0) \tag{3.6}$$

where

$$\overline{\gamma}(\beta, \mu^0):=\max\{\gamma(\beta, \mu^0),\ \mu^0\}. \tag{3.7}$$

Proof: (iii): From Proposition 3.1,

$$\begin{cases} -\gamma(\beta, \mu^0)\ <\ -\beta/2\ \leq\ \ \overline{y}_i\ \ \leq\ (1+4\kappa)\gamma(\beta, \mu^0), \\ -(1+4\kappa)\gamma(\beta, \mu^0)\ \leq\ -\overline{x}_i\ \ \leq\ \beta/2\ <\ \gamma(\beta, \mu^0). \end{cases}$$

Thus, we have

$$-2(1+2\kappa)\gamma(\beta, \mu^0)\leq\ \overline{y}_i-\overline{x}_i\ \leq\ 2(1+2\kappa)\gamma(\beta, \mu^0).$$

Therefore, we obtain the lower bound of y'_i as follows.

$$\begin{aligned} y'_i=\overline{y}_i-\overline{\phi}_i/2 &=\frac{1}{2}\{(\overline{y}_i-\overline{x}_i)+\sqrt{(\overline{y}_i-\overline{x}_i)^2+4\overline{\mu}^2}\} \\ &=\frac{\overline{y}_i-\overline{x}_i}{2}+\sqrt{\left(\frac{\overline{y}_i-\overline{x}_i}{2}\right)^2+\overline{\mu}^2} \\ &\geq-(1+2\kappa)\gamma(\beta, \mu^0)+\sqrt{(1+2\kappa)^2\gamma(\beta, \mu^0)^2+\ \overline{\mu}^2} \\ &=\frac{\overline{\mu}^2}{(1+2\kappa)\gamma(\beta, \mu^0)+\sqrt{(1+2\kappa)^2\gamma(\beta, \mu^0)^2+\ \overline{\mu}^2}} \\ &\geq\frac{\overline{\mu}^2}{(1+2\kappa)\gamma(\beta, \mu^0)+(1+2\kappa)\sqrt{\gamma(\beta, \mu^0)^2+\ \overline{\mu}^2}} \\ &\geq\frac{\overline{\mu}^2}{(1+\sqrt{2})(1+2\kappa)\overline{\gamma}(\beta, \mu^0)}. \end{aligned}$$

The first inequality is depend on the fact that the function $g(a):=a+\sqrt{a^2+b}$ with some $b>0$ is strictly increasing with respect to $a\in I\!R$. Next, the upper bound of y'_i is as follows.

$$y'_i=\overline{y}_i-\overline{\phi}/2\ \ \leq\ \ (1+4\kappa)\gamma(\beta, \mu^0)+\frac{\beta}{2}\ \ <\ \ 2(1+2\kappa)\gamma(\beta, \mu^0). \tag{3.8}$$

By a similar discussion, both the lower and the upper bound of x'_i are obtained. ∎

By the Lemma 3.4 and 4.1 of [KMNY91] for the type of the system (3.5), the following results are obtained.

Proposition 3.3. *Suppose that M is an* $n\times n$ $P^*(\kappa)$*-matrix for some* $k\geq 0$. *For every* $(\boldsymbol{x}', \boldsymbol{y}')>(\mathbf{0}, \mathbf{0})$ *and* $\overline{\boldsymbol{h}}\in I\!R^n$, *the system* (3.5) *has the unique solution* $(\Delta\boldsymbol{x}, \Delta\boldsymbol{y})$ *which satisfies the following inequalities:*

$$\begin{cases} -\kappa\|(X'Y')^{-\frac{1}{2}}\overline{\boldsymbol{h}}\|^2\ \leq\ \Delta\boldsymbol{x}^T\Delta\boldsymbol{y}\ \leq\ \frac{1}{4}\|(X'Y')^{-\frac{1}{2}}\overline{\boldsymbol{h}}\|^2, \\ \\ \|(X'Y')^{-\frac{1}{2}}\Delta\boldsymbol{x}\|^2+\|(X'Y')^{-\frac{1}{2}}X'\Delta\boldsymbol{y}\|^2\ \leq\ (1+2\varkappa)\|(X'Y')^{-\frac{1}{2}}\overline{\boldsymbol{h}}\|^2 \end{cases} \tag{3.9}$$

for som $\kappa \geq 0$. ∎

Corollary 3.4. *Suppose that Condition 1.1 holds and let* $(\bar{\mu}, \bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in \mathbb{R}_{++} \times \mathbb{R}^{2n}$ *be in the set* $\Lambda(\beta, \mu^0)$ *for some* $\beta > 0$ *and* $\mu^0 > 0$. *Then*

$$\|\Delta \boldsymbol{x}\|^2 + \|\Delta \boldsymbol{y}\|^2 \leq \frac{9(1+2\kappa)^3 \bar{\gamma}(\beta, \mu^0)^2}{\bar{\mu}^4} \|\bar{\boldsymbol{h}}\|^2.$$

Proof: By Proposition 1.2 and the definition of $(\boldsymbol{x}', \boldsymbol{y}')$, we have $X'Y' = \bar{\mu}^2 I$ with $\bar{\mu} > 0$. Thus the second inequality of (3.9) can be rewritten as follows.

$$\|Y'\Delta \boldsymbol{x}\|^2 + \|X'\Delta \boldsymbol{y}\|^2 \leq (1+2\varkappa)\|\bar{\boldsymbol{h}}\|^2.$$

By (iii) of Proposition 3.2,

$$\|(X')^{-1}\| \leq \frac{3(1+2\kappa)\bar{\gamma}(\beta, \mu^0)}{\bar{\mu}^2}, \quad \|(Y')^{-1}\| \leq \frac{3(1+2\kappa)\bar{\gamma}(\beta, \mu^0)}{\bar{\mu}^2}.$$

Thus we have

$$\|\Delta \boldsymbol{x}\| \leq \|(Y')^{-1}\|\|Y'\Delta \boldsymbol{x}\| \leq \frac{3(1+2\kappa)\bar{\gamma}(\beta, \mu^0)}{\bar{\mu}^2} \|Y'\Delta \boldsymbol{x}\|, \tag{3.10}$$

$$\|\Delta \boldsymbol{y}\| \leq \|(X')^{-1}\|\|X'\Delta \boldsymbol{y}\| \leq \frac{3(1+2\kappa)\bar{\gamma}(\beta, \mu^0)}{\bar{\mu}^2} \|X'\Delta \boldsymbol{y}\|. \tag{3.11}$$

Then,

$$\begin{aligned}\|\Delta \boldsymbol{x}\|^2 + \|\Delta \boldsymbol{y}\|^2 &\leq \|(Y')^{-1}\|^2\|Y'\Delta \boldsymbol{x}\|^2 + \|(X')^{-1}\|^2\|X'\Delta \boldsymbol{y}\|^2 \\ &\leq \frac{9(1+2\varkappa)^2\bar{\gamma}(\beta, \mu^0)^2}{\bar{\mu}^4}(\|Y'\Delta \boldsymbol{x}\|^2 + \|X'\Delta \boldsymbol{y}\|^2).\end{aligned}$$

Therefore, the desired result is obtained. ∎

The last proposition gives a second order approximation of the behavior of $\boldsymbol{\Phi}$ along the Newton direction $(\Delta\mu, \Delta \boldsymbol{x}, \Delta \boldsymbol{y})$.

Proposition 3.5. (*Proposition 6 of* [*HIY00*]) *Let* $(\bar{\mu}, \bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in \mathbb{R}^{1+2n}$ *such that* $\boldsymbol{\Phi}(\bar{\mu}, \bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) = \bar{\boldsymbol{\Phi}} \leq 0$ *and let* $(\Delta\mu, \Delta \boldsymbol{x}, \Delta \boldsymbol{y})$ *be the solution of the system* (*2.1*).

(i) *For every* $i \in N$ *and* $\theta \in [0, 1]$,

$$\begin{aligned}0 &\geq \{1-\theta(1-\sigma_\phi)\}\bar{\phi}_i \\ &\geq \phi_i(\bar{\mu}+\theta\Delta\mu, \bar{\boldsymbol{x}}+\theta\Delta \boldsymbol{x}, \bar{\boldsymbol{y}}+\theta\Delta \boldsymbol{y}) \\ &\geq \{1-\theta(1-\sigma_\phi)\}\bar{\phi}_i - \frac{\theta^2}{\sigma_\mu\bar{\mu}}\{(1-\sigma_\mu)^2\bar{\mu}^2 + \Delta x_i^2 + \Delta y_i^2\}\end{aligned}$$

(ii)

$$\begin{aligned}&\|\boldsymbol{\Phi}(\bar{\mu}+\theta\Delta\mu, \bar{\boldsymbol{x}}+\theta\Delta \boldsymbol{x}, \bar{\boldsymbol{y}}+\theta\Delta \boldsymbol{y}\| \\ &\quad \leq \{1-\theta(1-\sigma_\phi)\}\|\bar{\boldsymbol{\Phi}}\| + \frac{\theta^2}{\sigma_\mu\bar{\mu}}\{(1-\sigma_\mu)^2\sqrt{n}\bar{\mu}^2 + \|\Delta \boldsymbol{x}\|^2 + \|\Delta \boldsymbol{y}\|^2\}.\end{aligned}$$ ∎

4 Complexity analysis of the algorithm for $P^*(\kappa)$-LCP

At first, we show the finite termination of Step 1.1 to derive a complexity bound of the algorithm. Throughout the discussions in this section, we assume that Condition 1.1 holds.

Lemma 4.1. (*Lemma 1 of* [*HIY00*]) *At each iteration k, the following inequality holds for any p in Step 1.1.*

$$\|\hat{\boldsymbol{\Phi}}^{p+1}\| \leq \max\left\{1 - \frac{\|\hat{\boldsymbol{\Phi}}^p\|\mu^k}{4(\|\Delta \boldsymbol{x}^p\|^2 + \|\Delta \boldsymbol{y}^p\|^2)}, \frac{1}{2}\right\}\|\hat{\boldsymbol{\Phi}}^p\|. \tag{4.1}$$

Lemma 4.2. (*c.f. Lemma 2 of* [*HIY00*])

(i) *Let* $(\tilde{\mu}, \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})$ *be any point generated in the algorithm. Then*

$$\boldsymbol{\Phi}(\tilde{\mu}, \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) \leq \mathbf{0}.$$

(ii) *Let* $(\tilde{\mu}^k, \tilde{\boldsymbol{x}}^k, \tilde{\boldsymbol{y}}^k)$ *be any point generated at the iteration* k. *Then*

$$\|\boldsymbol{\Phi}(\tilde{\mu}^k, \tilde{\boldsymbol{x}}^k, \tilde{\boldsymbol{y}}^k)\| \leq \beta^k$$

and hence

$$(\tilde{\mu}^k, \tilde{\boldsymbol{x}}^k, \tilde{\boldsymbol{y}}^k) \in \Lambda(\beta^k, \mu^0)$$

where

$$\beta^k := \frac{20(1+2\kappa)^3\overline{\gamma}(\epsilon, \mu^0)^2 n}{\mu^k}. \tag{4.2}$$

Proof: (i): By the construction of the algorithm, the value of ϕ_i $(i \in N)$ is always nonpositive.

(ii): In Step 1.2, we set $\sigma_\mu := 1/2$, $\sigma_\phi := 1$ and $\theta := 1$. Thus, by (ii) of Proposition 3.5, we have

$$\|\boldsymbol{\Phi}(\mu^{k+1}, \boldsymbol{x}^{k+1}, \boldsymbol{y}^{k+1})\| \leq \epsilon + \frac{2}{\mu^k}\left\{\frac{\sqrt{n}(\mu^k)^2}{4} + \|\Delta\boldsymbol{x}\|^2 + \|\Delta\boldsymbol{y}\|^2\right\}. \tag{4.3}$$

By Corollary 3.4, we see that

$$\begin{aligned}\|\Delta\boldsymbol{x}\|^2 + \|\Delta\boldsymbol{y}\|^2 &\leq \frac{9(1+2\kappa)^3\overline{\gamma}(\epsilon, \mu^0)^2}{(\mu^k)^4}(-(\mu^k)^2\sqrt{n})^2 \\ &= 9(1+2\kappa)^3\overline{\gamma}(\epsilon, \mu^0)^2 n.\end{aligned}$$

Since the definition of $\overline{\gamma}$ and $\varepsilon \leq \mu^k \leq \mu^0$, we obtain the bound

$$\begin{aligned}\|\boldsymbol{\Phi}(\mu^{k+1}, \boldsymbol{x}^{k+1}, \boldsymbol{y}^{k+1})\| &\leq \epsilon + \frac{2}{\mu^k}\left\{\frac{\sqrt{n}(\mu^k)^2}{4} + 9(1+2\kappa)^3\overline{\gamma}(\epsilon, \mu^0)^2 n\right\} \\ &\leq \epsilon + \frac{2}{\mu^k}\frac{37}{4}(1+2\kappa)^3\overline{\gamma}(\epsilon, \mu^0)^2 n \\ &\leq \frac{20(1+2\kappa)^3\overline{\gamma}(\epsilon, \mu^0)^2 n}{\mu^k}. \qquad \blacksquare\end{aligned}$$

Theorem 4.3. (*c.f. Theorem 1 of* [*HIY00*])

(i) *At each iteration* k, *Step1.1 terminates after* P^k *Newton iterations where*

$$P^k := \left\lceil 2\max\left\{\frac{72(1+2\kappa)^5\overline{\gamma}(\beta^k, \mu^0)^4\|\hat{\boldsymbol{\Phi}}^k\|}{(\mu^k)^5}, 1\right\}\log\frac{\|\boldsymbol{\Phi}^k\|}{\epsilon}\right\rceil.$$

(ii) *The total number of Newton iterations in the algorithm is bounded by*

$$\left\lceil \log\frac{\overline{\beta}}{\epsilon}\right\rceil \cdot \left\{\left\lceil \frac{4896(1+2\kappa)^5\overline{\gamma}(\overline{\beta}, \mu^0)^4\overline{\beta}}{\epsilon^5}\right\rceil + 3\left(\left\lceil \log\frac{\mu^0}{\epsilon}\right\rceil + 1\right)\right\}$$

where

$$\overline{\beta} = \frac{20(1+2\kappa)^3\overline{\gamma}(\epsilon, \mu^0)^2 n}{\epsilon}$$

Proof: (i): In Step 1.1, we set $\sigma_\mu := 1$ and $\sigma_\phi := 0$. By (ii) of Lemma 4.2, Corollary 3.4 and (iii) of Proposition 3.2, we have

$$\begin{aligned}\|\Delta x^p\|^2 + \|\Delta y^p\|^2 &\leq \frac{9(1+2\kappa)^3\overline{\gamma}(\beta^k, \mu^0)^2}{(\mu^k)^4}\|\boldsymbol{h}^p\|^2 \\ &\leq \frac{9(1+2\kappa)^3\overline{\gamma}(\beta^k, \mu^0)^2}{(\mu^k)^4}\frac{1}{4}\|X' + Y'\|^2\|\hat{\boldsymbol{\Phi}}^p\|^2\end{aligned}$$

$$\leq \frac{9(1+2\kappa)^3\overline{\gamma}(\beta^k,\mu^0)^2}{(\mu^k)^4}\frac{1}{4}16(1+2\kappa)^2\overline{\gamma}(\beta^k,\mu^0)^2\|\hat{\boldsymbol{\Phi}}^p\|^2$$
$$\leq \frac{36(1+2\kappa)^5\overline{\gamma}(\beta^k,\mu^0)^4}{(\mu^k)^4}\|\hat{\boldsymbol{\Phi}}^p\|^2$$

Since $\|\hat{\boldsymbol{\Phi}}^p\|\leq\|\hat{\boldsymbol{\Phi}}^0\|=\|\boldsymbol{\Phi}^k\|$ and the above inequality imply that

$$\begin{aligned}\frac{\|\hat{\boldsymbol{\Phi}}^p\|\mu^k}{4(\|\Delta\boldsymbol{x}^p\|^2+\|\Delta\boldsymbol{y}^p\|^2)} &\geq \frac{\mu^k\cdot(\mu^k)^4}{4\cdot 36(1+2\kappa)^5\overline{\gamma}(\beta^k,\mu^0)^4\|\hat{\boldsymbol{\Phi}}^p\|}\\ &= \frac{(\mu^k)^5}{144(1+2\kappa)^5\overline{\gamma}(\beta^k,\mu^0)^4\|\hat{\boldsymbol{\Phi}}^p\|}\\ &\geq \frac{(\mu^k)^5}{144(1+2\kappa)^5\overline{\gamma}(\beta^k,\mu^0)^4\|\hat{\boldsymbol{\Phi}}^k\|}\end{aligned}$$

Thus, $\|\hat{\boldsymbol{\Phi}}^p\|$ will be reduced at least by $1-\delta^k$ where

$$\delta^k := \min\left\{\frac{(\mu^k)^5}{144(1+2\kappa)^5\overline{\gamma}(\beta^k,\mu^0)^4\|\hat{\boldsymbol{\Phi}}^k\|},\ \frac{1}{2}\right\} \tag{4.4}$$

at each iteration p in Step 1.1. Now we can obtain a lower bound of p as a suffcient condition of p to satisfy $(1-\delta^k)^p\|\boldsymbol{\Phi}^k\|<\epsilon$, i.e.,

$$p \leq \left\lceil \frac{1}{\delta^k}\log\frac{\epsilon}{\|\boldsymbol{\Phi}^k\|}\right\rceil.$$

Therefore, the assertion follows from Lemma 4.1.

(ii):Since $\beta^k\leq\overline{\beta}$, $\overline{\gamma}(\beta^k,\mu^0)\leq\overline{\gamma}(\overline{\beta},\mu^0)$ and $\|\boldsymbol{\Phi}^k\|\leq\beta^k$ by (ii) of Lemma 4.2, we have that

$$P^k \leq \left\lceil\log\frac{\overline{\beta}}{\epsilon}\right\rceil\cdot\left\lceil 2\left(\frac{72(1+2\kappa)^5\overline{\gamma}(\overline{\beta},\mu^0)^4\overline{\beta}}{(\mu^k)^5}\right)\right\rceil$$

for every k. Then, by a similar discussion as in the proof of Theorem 1 in [HIY00], the Newton iteration is bounded by

$$\sum_{k=0}^{K}P^k \leq \left\lceil\log\frac{\overline{\beta}}{\epsilon}\right\rceil\cdot\left\{\left\lceil 144(1+2\kappa)^5\overline{\gamma}(\overline{\beta},\mu^0)^4\overline{\beta}\cdot\frac{34}{\epsilon^5}\right\rceil+3\left(\left\lceil\log\frac{\mu^0}{\epsilon}\right\rceil+1\right)\right\}. \quad \blacksquare$$

5 Concluding remarks

The complexity bound of the smoothing algorithm for $P^*(\kappa)$-LCP has been presented. The results in this paper extend the complexity analysis proposed for monotone LCP in [HIY00] to the one for $P^*(\kappa)$-LCP. Consequently, we have shown that the algorithm terminates in

$$O\left(\frac{\kappa^8\overline{\gamma}^6 n}{\epsilon^6}\log\frac{\overline{\gamma}\sqrt{n}}{\epsilon}\right)$$

Newton iterations where $\overline{\gamma}$ is a number which depends on the problem and the initial point, and κ is a positive constant which depends on the problem.

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