The Minimum-weight k-Ideal Problem on a Directed Tree Poset and Its Polyhedron

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Abstract

Suppose we are given a poset(partially ordered set) $\mathcal{P}=(V,\preceq)$, a real-valued weight $w(v)$ associated with each element $v\in V$ and a positive integer κ. We consider the problem which asks to find an ideal of size κ of $\mathcal{P}$ such that the sum of the weights of the elements of this ideal is the minimum for all ideals that can be constructed from $\mathcal{P}$. We call this problem the minimum-weight κ-ideal problem. In this paper we explore a further possibility for a solvable case of this problem because it has been proven that this problem is $\mathcal{NP}$-hard even if the Hasse diagram representing a given poset is a bipartite graph. We obtain two new results. First, we describe an $O(\kappa^2 n)$ algorithm on the special poset whose Hasse diagram is a directed tree, which is called a directed tree poset. Here n is the cardinality of the underlying set of the given poset. This outperforms an $O(n^4)$ algorithm that is the best previously known. Secondly, we show a characteristic of the polyhedron obtained from the LP-relaxation of the κ-ideal problem on a directed tree poset. This is the first step to solve the other type κ-ideal problems from the LP-relaxation technique.

1. Introduction

A binary relation $\preceq$ on a set V is called a *partial order* if it satisfies the following three conditions: $\forall v\in V$: $v\preceq v$(Reflexivity), $v\preceq w$, $w\preceq v \Rightarrow v=w$(Antisymmetry), $u\preceq v$, $v\preceq w \Rightarrow u\preceq w$(Transitivity). If $u\preceq v$ and $u\neq v$, we write $u\prec v$. We call the pair $(V,\preceq)$ a *partially ordered set* or a *poset* for short. A poset $\mathcal{P}=(V,\preceq)$ is usually represented by a (directed) graph $G(\mathcal{P})=(V, A(\mathcal{P}))$ which is called the *Hasse diagram*, where $A(\mathcal{P})=\{(u,v) \mid u, v\in V, u\prec v$ and there exists no element $\hat{v}$ such that $u\prec \hat{v}\prec v\}$. For a poset $\mathcal{P}=(V,\preceq)$ a subset I of V is called an *ideal* of P if $u\preceq v\in I$ always implies $u\in I$. Posets and iedals play a fundamental and essential role in the systems represented by a directed graph or a network. Therefore, they lie in numerous application settings and in many forms (see, e. g. , [1], [6], [7]).

Suppose we are given a poset $\mathcal{P}=(V,\preceq)$, a real-valued weight $w(v)$ associated with each element $v\in V$ and a positive integer κ. Then the *minimum-weight cardinality-restricted ideal problem* (of size κ) or the *minimum-weight k-ideal problem* discussed in this paper is formulated as follows:

$$\mathrm{P}_\kappa:\ \text{Minimize} \sum_{v\in I} w(v) \tag{1.1a}$$

$$\text{subject to } I \in \mathcal{I}(\mathcal{P}), \tag{1.1b}$$

$$|I|=\kappa, \tag{1.1c}$$

where $\mathcal{I}(\mathcal{P})$ is the set of all the ideal of $\mathcal{P}$. The optimization problem on the ideal is valuable because many application in real-life are formalized as the ideal problem such as the above problem. Therefore, various types of this class of problems have been well researched, and Problem P_κ is known to be $\mathcal{NP}$-hard to the size of the Hasse deagram $G(\mathcal{P})$ representing a given poset $\mathcal{P}$ (see[4] for the concept of $\mathcal{NP}$-hard) even if the Hasse diagram $G(\mathcal{P})$ is a bipartite graph (see[3] for basic trerminology on the graph theory). This proposition drives us to the question on which families of posets we can solve Problem P_κ in polynomial time.

A study on this question was recently made by Faigle and Kern[2], and it revealed that we can solve Problem P_κ in $O(n^4)$ on the special poset whose Hasse diagram is a directed tree. We call such a poset a *directed tree poset*. Essentially more general classes of posets having a polynomial time algorithm for this problem have not been found yet. It still remains to be done.

In this paper we focus on Problem P_κ on a directed tree poset. First, we propose a simple implementation of the $O(n^4)$ algorithm for Problem P_κ on a directed tree poset proposed by Faigle and Kern[2], and reduce its complexity to $O(\kappa^2 n)$. The main cause of improvement is to transform the directed tree poset into an auxiliary binary tree. It leads to a simple computation and a strict estimation of the time complexity. Secondly, we show characteristics of the polyhedron obtained from the LP-relaxation of the κ-ideal problem on a directed tree poset. This result is the first step to solve the other type κ-ideal problems from the LP-relaxation technique. Each of the results has been developed from that in [2].

2. An Improved Algorithm

In this section we propose an $O(\kappa^2 n)$ improved algorithm for the minimum-weight κ-ideal problem(P_κ) on a directed tree poset $\mathcal{T}=(V,\preceq)$ with root v_1. The essential behavior of the improved algorithm is the same as that of the algorithm proposed by[2]: For the Hasse diagram $G(\mathcal{T})=(V, A(\mathcal{T}))$ of $\mathcal{T}$, starting with the leaves of $G(\mathcal{T})$, we successively move down to the root by computing the optimal value for Problem P_t $(0 \leqq t \leqq \min\{|V(v)|, \kappa\})$ on the directed subtree with root v of $G(\mathcal{T})$. Here,

$$V(v)=\{u \mid \text{there is the path from } u \text{ to } v \text{ in } G(\mathcal{T})\}. \tag{2.1}$$

The original algorithm[2] takes $O(n^3)$ to compute the optimal value at a vertex from the values for its children if the number of them is more than 2. However, we improved the time complexity to do it to $O(\kappa^2)$ by transforming from $G(\mathcal{T})$ into an auxiliary binary tree, which is a binary directed tree, i. e., the number of children of any vertex is at most 2, and from whose directed subtree we can retrieve the ideal of $\mathcal{T}$ easily.

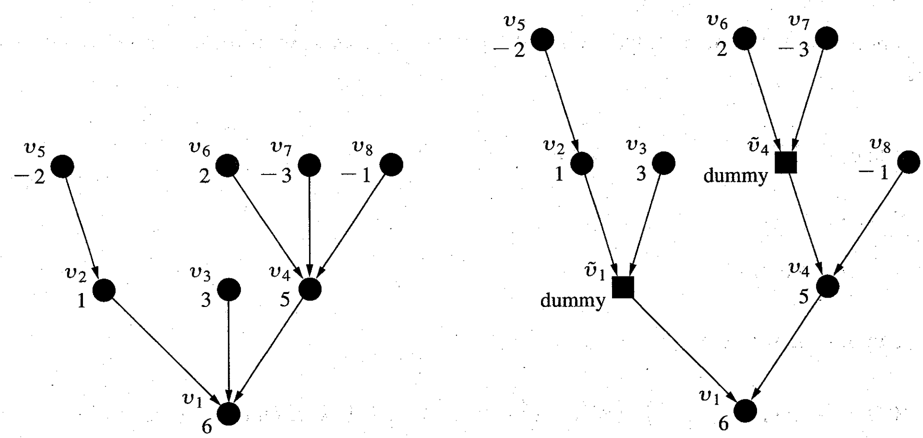
First, for the Hasse diagram $G(\mathcal{T})=(V, A(\mathcal{T}))$ of $\mathcal{T}$, we construct a directed tree $\tilde{G}(\mathcal{T})=(\tilde{V}, \tilde{A}(\mathcal{T}))$ by the following steps:

Step 1 : Put $\tilde{V}=V$ and $\tilde{A}(\mathcal{T})=A(\mathcal{T})$.

Step 2 : Repeat the following(*) until $\{v\in\tilde{V} \mid$ the number of v's children is more than $2\}=\emptyset$.

(*) Find a vertex $v\in\tilde{V}$ such that the number of v's children is more than 2 and choose its two children u_1, u_2. Add a dummy vertex $\tilde{v}$ into $\tilde{G}(\mathcal{T})$ such that its parent is v and its two child are u_1 and u_2. That is, put $\tilde{V} := \tilde{V} \cup \{\tilde{v}\}$ and $\tilde{A}(\mathcal{T}) := (\tilde{A}(\mathcal{T}) - \{(u, v)\}) \cup \{(u_1, \tilde{v}), (u_2, \tilde{v}), (\tilde{v}, v)\}$.

We call $\tilde{G}(\mathcal{T}) = (\tilde{V}, \tilde{A}(\mathcal{T}))$ obtrained by the above steps an *auxiliary tree* for $\mathcal{T}$. See Figure 1 for an example of the auxiliary tree $\tilde{G}(\mathcal{T})$ for a directed tree poset $\mathcal{T}$.



(a) The Hasse diagram $G(\mathcal{T})$. (b) The auxiliary tree $\tilde{G}(\mathcal{T})$ for $G(\mathcal{T})$.

Figure 1 : An example for a directed tree poset $\mathcal{T}$.

We remark that the auxiliary tree $\tilde{G}(\mathcal{T})$ has the following features:

(1) The root v_1 and leaves are not dummy vertex;

(2) For a directed subtree $\hat{\mathcal{T}} = (\hat{\mathcal{T}}, \hat{A})$ with root v_1 of $\tilde{G}(\mathcal{T})$ and the set V^d of all the dummy vertices in $\hat{V}$, the set $\hat{V} - V^d$ is an ideal of $\mathcal{T}$;

(3) $|\hat{V}| \leqq 2|V| - 3$ and $|\tilde{A}(\mathcal{T})| \leqq 2|V| - 4$.

Algorithm

Input : A directed tree poset $\mathcal{T} = (V, \preceq)$ withroot v_1 and a positive integer κ.

Output : The solution of Problem P_κ on $\mathcal{T}$.

Step 1 : Construct the auxiliary tree $\tilde{G}(\mathcal{T}) = (\tilde{V}, \tilde{A}(\mathcal{T}))$ for $\mathcal{T}$.

Step 2 : Repeat the following(*) for v_i in topological order.

(*) Compute $W(t, v_i)$ $(0 \leqq t \leqq \min\{|\tilde{V}(v_i)|, \kappa\})$ according to the above(i), (ii) and (iii).

Step 3 : The current $W(\kappa, v_1)$ is the optimal solution.

(End)

Figure 2 : The algorithm for determining $W(\kappa, v_1)$.

Secondly, we sketch an implementation using $\tilde{G}(\mathcal{T})$. For a vertex $v_i \in \tilde{V}$ $\hat{T}(v_i)=(\hat{V}(v_i), \hat{A}(v_i))$ stands for a directed subtree with root v_i of $\tilde{G}(\mathcal{T})$ and $\hat{V}^d(v_i)$ the set of all the dummy vertices in $\hat{V}(v_i)$. Furthermore, let us define

$$W(t, v_i) = \min \left\{ \sum_{v \in \hat{V}(v_i)} w(v) \;\middle|\; |\hat{V}(v_i) - \hat{V}^d(v_i)| = t \right\} \quad (0 \leqq t \leqq \min\{|\tilde{V}(v_i)|, \kappa\}). \tag{2.2}$$

We can compute $W(t, v_i)$ from the values obtained for at most two children of v_i, which is denoted by $u^1_{v_i}$ and $u^2_{v_i}$, as follows:

(i) For a leaf v_i of $\mathcal{T}$

$$W(0, v_i) = 0, \tag{2.3a}$$

$$W(1, v_i) = w(v_i). \tag{2.3b}$$

(ii) For a dummy inner vertex v_i

$$W(t, v_i) = \min\{W(t', u^1_{v_i}) + W(t'', u^2_{v_i}) \mid t' + t'' = t\} \quad (0 \leqq t \leqq \min\{|\tilde{V}(v_i)|, \kappa\}). \tag{2.4}$$

(iii) For a non-dummy inner vertex v_i

$$W(0, v_i) = 0, \tag{2.5a}$$

$$W(t, v_i) = w(v_i) + \min\{W(t', u^1_{v_i}) + W(t'', u^2_{v_i}) \mid t' + t'' = t - 1\} \quad (1 \leqq t \leqq \min\{|\tilde{V}(v_i)|, \kappa\}). \tag{2.5b}$$

In the case when an inner vertex v_i has only one child $u^1_{v_i}$, replace $\min\{W(t', u^1_{v_i}) + W(t'', u^2_{v_i}) \mid t'+t''=t\}$ in (2.4) by $W(t, u^1_{v_i})$, and $\min\{W(t', u^1_{v_i}) + W(t'', u^2_{v_i}) \mid t'+t''=t-1\}$ in (2.5b) by $W(t-1, u^1_{v_i})$.

The objective function value of an optimal solution of Problem P_κ on $\mathcal{T}$ is given by $W(\kappa, v_1)$ due to the definition. The above computation method of $W(t, v_i)$ suggests the "from leaves to root" algorithm for determining $W(\kappa, v_1)$ as shown in Figure 2.

Also see Figure 3 for illustrating the algorithm in the case of $\kappa=4$ on the directed tree poset given in Figure 1. We have $W(4, v_1)=7$ and the corresponding set $\{v_1, v_4, \tilde{v}_4, v_7, v_8\}$; hence, the minimum-wight ideal of size 4 of $\mathcal{T}$ is $\{v_1, v_4, v_7, v_8\}$.

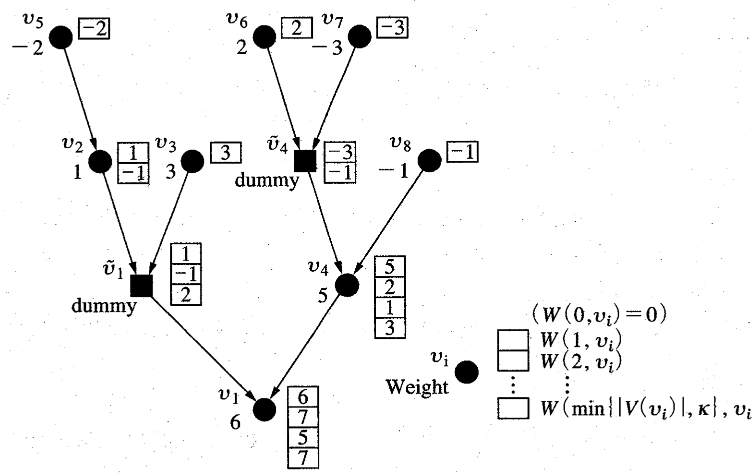


Figure 3 : Illustrating the algorithm in the case of $\kappa=4$ on the derected tree poset T shown in Figure 1.

Theorem 2.1 : *The algorithm shown in Figure 2 computes $W(\kappa, v_1)$ in $O(\kappa^2 n)$ time.*

(Proof) The correctness of the algorithm is clear from the above discussion. We consider the time complexity for the algorithm. Constructing the auxiliary tree $\tilde{G}(T)=(\tilde{V}, \tilde{A}(T))$ takes $O(n)$ time. And we need $O(\kappa^2)$ time to compute $W(t, v_i)(0 \leqq t \leqq \min\{|\tilde{V}(v_i)|, \kappa\})$ at a vertex v_i. Because, in an iteration fo Step 2, the number of all the pairs of two numbers such that one is chose from $\{0, 1, \cdots, \min\{|\tilde{V}(u_{v_i}^1)|, \kappa\}\}$ and the other is chosen from $\{0, 1, \cdots, \min\{|\tilde{V}(u_{v_i}^2)|, \kappa\}\}$ is at most $(\kappa+1)^2$, and the other computations take a constant time.

The number of iterations of Step 2 is $|\tilde{V}|$ times. Notice that $|\tilde{V}| \leqq 2|V|-3$. Hence, the overall bound is $O(\kappa^2 n)$. □

If an optimal ideal is required, then the additional data should be held all the contributing vertices in computing $W(t, v_i)$.

3. A Characteristic of the Polyhedron

In this section we show a property of the polyhedron obtained from the LP-relaxation of the cardinality-restricted ideal problem P_κ on a directed tree poset.

The LP-relaxation of Problem P_κ on a general poset is given as follows:

$$\mathrm{P}_{\kappa}^{\mathrm{LP}}: \text{ Minimize } \boldsymbol{wx} \tag{3.1a}$$

$$\text{subject to } D^t \boldsymbol{x} \leqq \mathbf{0}_m, \tag{3.1b}$$

$$\sum_{v \in V} x(v) = \kappa, \tag{3.1c}$$

$$\mathbf{0}_n \leqq \boldsymbol{x} \leqq \mathbf{1}_n, \tag{3.1d}$$

where D^t denotes the transposition of the incidence matrix D of the Hasse diagram representing a given poset, and $\mathbf{0}_n(\mathbf{1}_n)$ stands for the all-0 (1) column vector of size n.

It is well known the following fundamental relationship between the non-cardinality-restricted ideal problems which is formulated as min $\{\Sigma_{v\in I} w(v) \mid I \in \mathcal{I}(\mathcal{P})\}$ and its LP-relaxation, denoted by P^{LP}, whose constraint system do not have only (3.1c) in the above $\mathrm{P}_{\kappa}^{\mathrm{LP}}$.

Proposition 3.1(see [8]) : *The polyhedron $Q = \{x \mid D^t x \leqq \mathbf{0}_m,\ \mathbf{0}_n \leqq x \leqq \mathbf{1}_n\}$ is integral, i. e., min$\{wx \mid x \in Q\}$ is attained by an integral vector for each w for which the minimum is finite.* □

See Figure 4 for an example of a polyhedron Q defined in Proposition 3.1.

Propositon 3.2(see [8]) : *An integral optimal solution of* Problem P^{LP} *is an optimal solution of the minimum-weight non-cardinality-restricted ideal problem.* □

Due to Proposition 3.2 we can obtain an optimal ideal for the minimum-weight non-cardinality-restricted ideal problem by applying methods for the linear programming problem such as the simplex method.

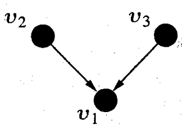
However, the additional constraint, $\Sigma_{v\in V} x(v) = \kappa$, breaks the above property, i. e ., the polyhedron

$$Q_\kappa = \left\{ x \in Q \;\middle|\; \sum_{v \in V} x(v) = \kappa \right\} \tag{3.2}$$

is not integral, where the set Q is defined in Proposition 3.1. (See Figure 5 for an example of such polyhedrons.) Therefore, we get difficult in solving the κ-ideal problem from LP-relaxation technique by this observation even if a given poset is a directed tree poset. Nevertheless, we can see the particular structure of the polyhedron Q_κ on a directed tree poset.

Lemma 3.3 : *For a directed tree poset $\mathcal{T} = (V, \preceq)$ with root v_1 and a positive integer κ with $\kappa \leqq |V|$, consider the polyhedron Q_κ defined by* (3.2). *For a vertex $\hat{x}$ of Q_κ let us define sets*

$$V_1 = \{v \in V \mid \hat{x}(v) = 1\} \tag{3.3}$$



A poset T

The Polyhedron Q

$$Q=\left\{x=\begin{pmatrix}x(v_1)\\x(v_2)\\x(v_3)\end{pmatrix}\in R^3 \;\middle|\; \begin{matrix}\begin{pmatrix}-1&1&0\\-1&0&1\end{pmatrix}x\leqq\begin{pmatrix}0\\0\end{pmatrix},\\ 0\leqq x(v_1),\ x(v_2), x(v_3)\leqq 1.\end{matrix}\right\}$$

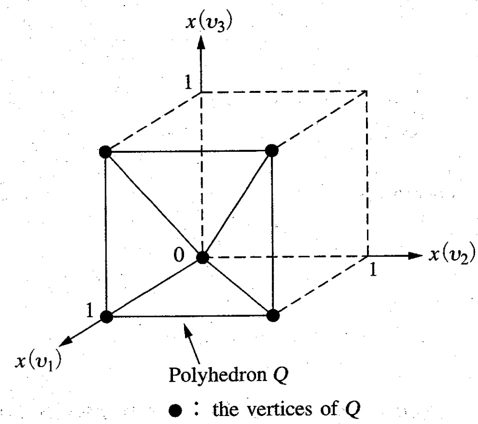


Figure 4 : An example of a poset T and its polyhedron Q.

and

$$V_2=\{v\in V \mid 0<\hat{x}(v)<1\}. \tag{3.4}$$

Then, for all $v\in V_2$ $\hat{x}(v)$ has the same constant value. Indeed, for all $v\in V_2$

$$\hat{x}(v)=\frac{\kappa-|V_1|}{|V_2|}. \tag{3.5}$$

(Proof) If $\hat{\boldsymbol{x}}$ is a vertex of Q_κ, then it is determined by n linearly independent equations from the systm given by (3.1b)~(3.1d) (see[8]), which are expressed as follows:

$$D_0^t\hat{\boldsymbol{x}}=\mathbf{0}_{|V_2|-1} \text{ for submatrix } D_0 \text{ of } D, \tag{3.6a}$$

$$\mathbf{1}_n^t\hat{\boldsymbol{x}}=\kappa, \tag{3.6b}$$

$$\hat{x}(v)=1\ (v\in V_1), \tag{3.6c}$$

$$\hat{x}(v)=0\ (v\in V_3), \tag{3.6d}$$

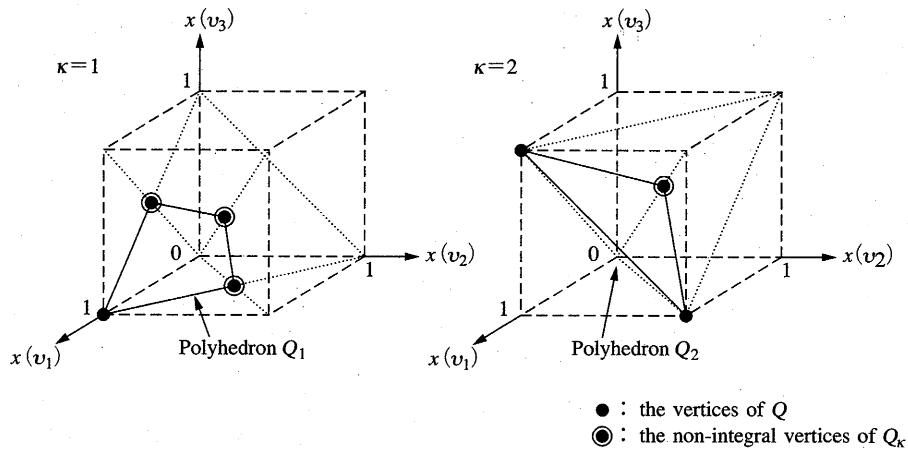


Figure 5 : The Polyhedron Q_κ(κ=1, 2) for the directed tree poset T shown in Figure 4.

where $V_3 \equiv V-(V_1 \cup V_2)=\{v \in V \mid \hat{x}(v)=0\}$. Notice that D_0^t can be expressed as

$$D_0^t = \begin{pmatrix} D_{V_1}^t & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & D_{V_2}^t & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & D_{V_3}^t \end{pmatrix}, \tag{3.7}$$

where $D_{V_i}(i=1, 2, 3)$ denotes the incidence matrix of the subgraph of the Hasse diagram $G(T)$ of the given directed tree poset T induced by V_i, respectively. By deleting the evident factor from the above n linearly independent equations we obtain the following equations with respect to $\hat{x}(V_2)=(x(v) \mid v \in V_2)$:

$$D_{V_2}^t \hat{x}(V_2) = \mathbf{0}_{|V_2|-1}, \tag{3.8a}$$

$$\mathbf{1}_{|V_2|}^t \hat{x}(V_2) = \kappa - |V_1|. \tag{3.8b}$$

These are linearly independent, that is,

$$\operatorname{rank}\begin{pmatrix} D_{V_2}^t \\ \mathbf{1}_{|V_2|}^t \end{pmatrix} = |V_2|. \tag{3.9}$$

Therefore,

$$\hat{x}(V_2) = \begin{pmatrix} D_{V_2}^t \\ \mathbf{1}_{|V_2|}^t \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{0}_{|V_2|-1} \\ \kappa-|V_1| \end{pmatrix} = \frac{1}{|V_2|}\begin{pmatrix} * & \begin{matrix} 1 \\ \vdots \\ 1 \end{matrix} \end{pmatrix}\begin{pmatrix} \mathbf{0}_{|V_2|-1} \\ \kappa-|V_1| \end{pmatrix} = \frac{\kappa-|V_1|}{|V_2|}\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}. \tag{3.10}$$

We have thus proved the lemma. □

Lemma 3.4 : *For the set V_1 and V_2 defined in* Lemma 3.3 *let us define subgraphs $T_1=(V_1, A_{V_1})$ and $T_2=(V_2, A_{V_2})$ of the Hasse diagram $G(\mathcal{T})$ of $\mathcal{T}$ induced by V_1 and V_2, respectively. Then T_1 and T_2 are the directed subtrees of $G(\mathcal{T})$ satisfying the following three conditions:*

(i) *The root of T_1 is v_1.*
(ii) *The parent $\hat{v}$ of the root v' of T_2 is in V_1 if $V_1 \neq \emptyset$; otherwise $v'=v_1$.*
(iii) $|V_1|+|V_2| \geqq \kappa$ *and* $V_1 \cap V_2=\emptyset$.

(Proof) It is clear that the condition(iii) holds true. T_1 is a directed subtree with root v_1 since the set V_1 forms an ideal of $\mathcal{T}$ and v_1 is the unique minimal element of $\mathcal{T}$. From (3.9) we have

$$\mathrm{rank} D_{V_2}^{t}=|V_2|-1. \tag{3.11}$$

This implies that T_2 forms a tree (see[5]), and a subtree T_2 of $G(\mathcal{T})$ becomes a directed tree. For the root v' of T_2 and the parent $\hat{v}$ of v', the constraint (3.1b) and the fact that $\hat{v} \notin V_2$ impose an inequality $x(\hat{v})>x(v')$. By virtue of Lemma 3.3 the non-integral value of $\hat{x}$ is unique. Hence, $x(\hat{v})$ must be 1, i. e. , $\hat{v} \in V_1$. If $V_1=0$, then $v'=v_1$ since the set V_2 must form an ideal of $\mathcal{T}$ instead of V_1. □

Lemma 3.5 : *Consider two directed subtrees $T_1=(V_1, A_{V_1})$ and $T_2=(V_2, A_{V_2})$ of the Hasse diagram $G(\mathcal{T})$ of a directed tree poset $\mathcal{T}=(V, \preceq)$ with root v_1 satisfying the conditions shown in* Lemma 3.4 *for a positive integer κ with $\kappa \leqq |V|$. Let us define $\hat{x}$ by*

$$\hat{x}(v)=\begin{cases} 1 & (\text{if } v \in V_1) \\ \frac{\kappa-|V_1|}{|V_2|} & (\text{if } v \in V_2) \\ 0 & (\text{otherwise}). \end{cases} \tag{3.12}$$

Then, $\hat{x}$ is a vertex of the polyhedron Q_κ.

(Proof) We can see that $\hat{x} \in Q_\kappa$ by substituting $\hat{x}$ into(3.1b)~(3.1d). Now, we show that $\hat{x}$ is a vertex of Q_κ. Some active inequalities for $\hat{x}$ are listed below. (Here, we say an inequality $\boldsymbol{a}^t\boldsymbol{x} \leqq \boldsymbol{b}$ is active for $\hat{x}$ in a given system if $\boldsymbol{a}^t\hat{x}=\boldsymbol{b}$.)

$$\begin{pmatrix} D_{V_1}^t & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & D_{V_2}^t & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & D_{V_3}^t \\ 1\cdots1 & 1\cdots1 & 1\cdots1 \\ E_{|V_1|} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & E_{|V_3|} \end{pmatrix} \hat{x}=\begin{pmatrix} \mathbf{0}n \\ \kappa \\ \mathbf{1}_{|V_1|} \\ \mathbf{0}_{|V_3|} \end{pmatrix} \tag{3.13}$$

The left side matrix of (3.13) is denoted by D'. Then, we have

$$\mathrm{rank} D' \geqq |V_1|+|V_2|+|V_3|=n \tag{3.14}$$

since each of the first and the third column of D' contributes at least $|V_1|$ and $|V_3|$ to the rank of D' and we have

$$\text{rank}\begin{pmatrix} D^t_{V_2} \\ 1\cdots 1 \end{pmatrix} = |V_2| \tag{3.15}$$

(see[5]). On the contrary, it is clear that rank$D' \leqq n$. Consequently, we have rank$D' = n$. It is known that the matrix D' has full column rank if and only if $\hat{x}$ is a vertex of Q_K, thereby completing the proof. (See[5],[8],[9] for a full account of the properties of the polyhedron.) □

Combining Lemma 3.4 with Lemma 3.5, we have the theorem below.

Theorem 3.6 : *The correspondence* (3.12) *gives a one-to-one and onto mapping from the set of all the pairs of the subtrees T_1 and T_2 of $G(T)$ satisfying the conditions in* Lemma 3. 4 *to the set of all the vertices of Q_K.* □

We can characterize the structure of Q_K for a directed tree poset. However, no result sheds some light on it for more general posets. Besides, it remains an unsettled question how to delete the non-integral vertex from Q_K. The LP-relaxation technique may not be suit for the cardinality-restricted ideal problems such as Problem P_K. Much still remains to be done.

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