Balanced Bowtie and Trefoil Decomposition of Symmetric Complete Tripartite Digraphs

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Abstract

First, we show that the necessary and sufficient condition for the existence of a balanced bowtie decomposition of the symmetric complete tripartite digraph $K^*_{n_1,n_2,n_3}$ is $n_1=n_2=n_3\equiv 0 \pmod 3$.
Next, we show that the necessary and sufficient condition for the existence of a balanced trefoil decomposition of the symmetric complete tripartite digraph $K^*_{n_1,n_2,n_3}$ is $n_1=n_2=n_3\equiv 0 \pmod 9$.

Keywords: Balanced bowtie decomposition; Balanced trefoil decomposition; Symmetric complete tripartite digraph

1. Introduction

Let $K^*_{n_1,n_2,n_3}$ denote *the symmetric complete tripartite digraph* with partite sets V_1, V_2, V_3 of n_1, n_2, n_3 vertices each. *The t-foil* (or *the t-windmill*) is a graph of t edge-disjoint triangles with a common vertex and the common vertex is called *the center of the t-foil.* In particular, the 2-foil and the 3-foil are called *the bowtie* and *the trefoil*, respectively. When $K^*_{n_1,n_2,n_3}$ is decomposed into edge-disjoint sum of t-foils, we say that $K^*_{n_1,n_2,n_3}$ *has a t-foil decomposition.* Moreover, when every vertex of $K^*_{n_1,n_2,n_3}$ appears in the same number of t-foils, we say that $K^*_{n_1,n_2,n_3}$ *has a balanced t-foil decomposition* and this number is called *the replication number*.

In Section 2, it is shown that the necessary and sufficient condition for the existence of a balanced bowtie decomposition of $K^*_{n_1,n_2,n_3}$ is $n_1=n_2=n_3\equiv 0 \pmod 3$.

In Section 3, it is shown that the necessary and sufficient condition for the existence of a balanced trefoil decomposition of $K^*_{n_1,n_2,n_3}$ is $n_1=n_2=n_3\equiv 0 \pmod 9$.

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Let K_n denote *the complete graph* of n vertices. And let C_3 be *the cycle* on 3 vertices. Then it is a well-known result that K_n has a C_3 - decomposition if and only if $n \equiv 1$ or 3 (mod 6). This decomposition is known as *a Steiner triple system.* See Wallis[2, Chapter 12 Triple Systems]. Horák and Rosa[1] proved that K_n has a bowtie-decomposition if and only if $n \equiv 1$ or 9 (mod 12). This decomposition is known as *a bowtie system.*

2. Balanced bowtie decomposition of $K^*_{n1,n2,n3}$

Notation. We denote a bowtie passing through $v_1 - v_2 - v_3 - v_1 - v_4 - v_5 - v_1$ by $\{(v_1,v_2,v_3),(v_1,v_4,v_5)\}$.

We have the following theorem.

Theorem 1. If $K^*_{n1,n2,n3}$ has a balanced bowtie decomposition, then $n_1 = n_2 = n_3 \equiv 0$ (mod 3).

Proof. Suppose that $K^*_{n1,n2,n3}$ has a balanced bowtie decomposition. Let b be the number of bowties and r be the replication number. Then $b = (n_1n_2 + n_1n_3 + n_2n_3)/3$ and $r = 5(n_1n_2 + n_1n_3 + n_2n_3)/3(n_1 + n_2 + n_3)$. Among r bowties having vertex v in V_i, ler r_{ij} be the number of bowties in which the centers are in V_j. Then $r_{11} + r_{12} + r_{13} = r_{21} + r_{22} + r_{23} = r_{31} + r_{32} + r_{33} = r$. Counting the number of vertices adjacent to vertex v in V_1, $2r_{11} + r_{12} + r_{13} = 2n_2$ and $2r_{11} + r_{12} + r_{13} = 2n_3$. Counting the number of vertices adjacent to vertex v in V_2, $r_{21} + 2r_{22} + r_{23} = 2n_1$ and $r_{21} + 2r_{22} + r_{23} = 2n_3$. Counting the number of vertices adjacent to vertex v in V_3, $r_{31} + r_{32} + 2r_{33} = 2n_1$ and $r_{31} + r_{32} + 2r_{33} = 2n_2$. Therefore, $n_1 = n_2 = n_3$. Put $n_1 = n_2 = n_3 = \mathrm{n}$. Then $b = n^2$, $r = 5n/3$, $r_{11} = r_{22} = r_{33} = n/3$ and $r_{12} + r_{13} = r_{21} + r_{23} = r_{31} + r_{32} = 4\mathrm{n}/3$. Thus $n \equiv 0$ (mod 3).
Therefore, $n_1 = n_2 = n_3 \equiv 0$ (mod 3) is necessary.

Let $\vec{K}_{s,s,s}$ denote *the complete tripartite digraph* with partite sets V_1, V_2, V_3 of s vertices each such that the directions of edges are directed away from V_1 to V_2, from V_2 to V_3, and from V_3 to V_1. *The* $(2)^s$*-foil* is a digraph of 2 edge-disjoint $\vec{K}_{s,s,s}$ with s common vertices and the common vertices are called *the centers of the* $(2)^s$*-foil.*

We have the following theorems.

Theorem 2. The $(2)^s$-foil has a balanced bowtie decomposition.

Proof. Denote the $(2)^s$-foil as $\{(V_1,V_2,V_3),(V_1,V_4,V_5))\}$, where $V_i = \{(i-1)s+1,(i-1)s+2,...,is\}$ $(i = 1,2,...,5)$.
Construct s^2 bowties B_{ij} $(i = 1,2,...,s; j = 1,2,...,s)$ as following:
$B_{11} = \{(1,s+1,2s+1),(1,3s+1,4s+1)\}$
$B_{12} = \{(1,s+2,2s+2),(1,3s+2,4s+2)\}$
...
$B_{1s} = \{(1,2s,3s),(1,4s,5s)\}$

$B_{21}=\{(2,s+1,2s+2),(2,3s+1,4s+2)\}$
$B_{22}=\{(2,s+2,2s+3),(2,3s+2,4s+3)\}$
...
$B_{2s}=\{(2,2s,2s+1),(2,4s,4s+1)\}$

...

$B_{s1}=\{(s,s+1,3s),(s,3s+1,5s)\}$
$B_{s2}=\{(s,s+2,2s+1),(s,3s+2,4s+1)\}$
...
$B_{ss}=\{(s,2s,3s-1),(s,4s,5s-1)\}$.
Then they comprise a balanced bowtie decomposition of the $(2)^s$-foil.

Theorem 3. If $K^*_{n,n,n}$ has a balanced bowtie decomposition, then $K^*_{sn,sn,sn}$ has a balanced bowtie decomposition.

Proof. When $K^*_{n,n,n}$ has a balanced bowtie decomposition, $K^*_{sn,sn,sn}$ has a balanced $(2)^s$-foil decomposition. By Theorem 2, the $(2)^s$-foil has a balanced bowtie decomposition. Therefore, $K^*_{sn,sn,sn}$ has a balanced bowtie decomposition.

Theorem 4. When $n\equiv 0 \pmod 3$, $K^*_{n,n,n}$ has a balanced bowtie decomposition.

Proof. Put $n=3s$. When $s=1$, let $V_1=\{1,2,3\}$, $V_2=\{4,5,6\}$, $V_3=\{7,8,9\}$.
Construct 9 bowties B_i $(i=1,2,\ldots,9)$ as following:
$B_1=\{(1,5,8),(1,9,6)\}$
$B_2=\{(2,6,9),(2,7,4)\}$
$B_3=\{(3,4,7),(3,8,5)\}$
$B_4=\{(4,8,2),(4,3,9)\}$
$B_5=\{(5,9,3),(5,1,7)\}$
$B_6=\{(6,7,1),(6,2,8)\}$
$B_7=\{(7,2,5),(7,6,3)\}$
$B_8=\{(8,3,6),(8,4,1)\}$
$B_9=\{(9,1,4),(9,5,2)\}$.
Then they comprise a balanced bowtie decomposition of $K^*_{3,3,3}$.
Applying Theorem 3, $K^*_{n,n,n}$ has a balanced bowtie decomposition.

Therefore, we have the following theorem.

Main Theorem 1. $K^*_{n_1,n_2,n_3}$ has a balanced bowtie decomposition if and only if $n_1=n_2=n_3\equiv 0 \pmod 3$.

3. Balanced trefoil decomposition of $K^*_{n1,n2,n3}$

Notation. We denote a trefoil passing through $v_1-v_2-v_3-v_1-v_4-v_5-v_1-v_6-v_7-v_1$ by $\{(v_1,v_2,v_3),(v_1,v_4,v_5),(v_1,v_6,v_7)\}$.

We have the following theorem.

Theorem 5. If $K^*_{n1,n2,n3}$ has a balanced trefoil decomposition, then $n_1=n_2=n_3\equiv 0 \pmod 9$.

Proof. Suppose that $K^*_{n1,n2,n3}$ has a balanced trefoil decomposition. Let b be the number of trefoils and r be the replication number. Then $b=2(n_1n_2+n_1n_3+n_2n_3)/9$ and $r=14(n_1n_2+n_1n_3+n_2n_3)/9(n_1+n_2+n_3)$. Among r trefoils having vertex v in V_i, ler r_{ij} be the number of trefoils in which the centers are in V_j. Then $r_{11}+r_{12}+r_{13}=r_{21}+r_{22}+r_{23}=r_{31}+r_{32}+r_{33}=r$. Counting the number of vertices adjacent to vertex v in V_1, $3r_{11}+r_{12}+r_{13}=2n_2$ and $3r_{11}+r_{12}+r_{13}=2n_3$. Counting the number of vertices adjacent to vertex v in V_2, $r_{21}+3r_{22}+r_{23}=2n_1$ and $r_{21}+3r_{22}+r_{23}=2n_3$. Counting the number of vertices adjacent to vertex v in V_3, $r_{31}+r_{32}+3r_{33}=2n_1$ and $r_{31}+r_{32}+3r_{33}=2n_2$. Therefore, $n_1=n_2=n_3$. Put $n_1=n_2=n_3=n$. Then $b=2n^2/3$, $r=14n/9$, $r_{11}=r_{22}=r_{33}=2n/9$ and $r_{12}+r_{13}=r_{21}+r_{23}=r_{31}+r_{32}=4n/3$. Thus $n\equiv 0 \pmod 9$.
Therefore, $n_1=n_2=n_3\equiv 0 \pmod 9$ is necessary.

Let $\vec{K}_{s,s,s}$ denote *the complete tripartite digraph* with partite sets V_1, V_2, V_3 of s vertices each such that the directions of edges are directed away from V_1 to V_2, from V_2 to V_3, and from V_3 to V_1. *The* $(3)^s$*-foil* is a digraph of 3 edge-disjoint $\vec{K}_{s,s,s}$ with s common vertices and the common vertices are called *the centers of the* $(3)^s$*-foil.*

We have the following theorems.

Theorem 6. The $(3)^s$-foil has a balanced trefoil decomposition.

Proof. Denote the $(3)^s$-foil as $\{(V_1,V_2,V_3), (V_1,V_4,V_5), (V_1,V_6,V_7)\}$, where $V_i=\{(i-1)s+1, (i-1)s+2,..., is\}$ $(i=1,2,...,7)$.
Construct s^2 trefoils B_{ij} $(i=1,2,...,s; j=1,2,...,s)$ as following:

$B_{11}=\{(1,s+1,2s+1),(1,3s+1,4s+1),(1,5s+1,6s+1)\}$

$B_{12}=\{(1,s+2,2s+2),(1,3s+2,4s+2),(1,5s+2,6s+2)\}$

...

$B_{1s}=\{(1,2s,3s),(1,4s,5s),(1,6s,7s)\}$

$B_{21}=\{(2,s+1,2s+2),(2,3s+1,4s+2),(2,5s+1,6s+2)\}$

$B_{22}=\{(2,s+2,2s+3),(2,3s+2,4s+3),(2,5s+2,6s+3)\}$

...

$B_{2s}=\{(2,2s,2s+1),(2,4s,4s+1),(2,6s,6s+1)\}$

...

$B_{s1}=\{(s,s+1,3s),(s,3s+1,5s),(s,5s+1,7s)\}$

$B_{s2}=\{(s,s+2,2s+1),(s,3s+2,4s+1),(s,5s+2,6s+1)\}$

...

$B_{ss}=\{(s,2s,3s-1),(s,4s,5s-1),\ldots,(s,6s,7s-1)\}$.

Then they comprise a balanced trefoil decomposition of the $(3)^s$-foil.

Theorem 7. If $K^*_{n,n,n}$ has a balanced trefoil decomposition, then $K^*_{sn,sn,sn}$ has a balanced trefoil decomposition.

Proof. When $K^*_{n,n,n}$ has a balanced trefoil decomposition, $K^*_{sn,sn,sn}$ has a balanced $(3)^s$-foil decomposition. By Theorem 6, the $(3)^s$-foil has a balanced trefoil decomposition. Therefore, $K^*_{sn,sn,sn}$ has a balanced trefoil decomposition.

Theorem 8. When $n\equiv 0 \pmod 9$, $K^*_{n,n,n}$ has a balanced trefoil decomposition.

Proof. Put $n=9s$. When $s=1$, let $V_1=\{1,2,\ldots,9\}$, $V_2=\{10,11,\ldots,18\}$, $V_3=\{19,20,\ldots,27\}$.

Construct 27 trefoils B_i $(i=1,2,\ldots,27)$ as following:

$B_1=\{(1,10,19),(1,11,20),(1,12,21)\}$

$B_2=\{(2,10,20),(2,11,21),(2,12,19)\}$

$B_3=\{(3,10,21),(3,11,19),(3,12,20)\}$

$B_4=\{(4,13,22),(4,14,23),(4,15,24)\}$

$B_5=\{(5,13,23),(5,14,24),(5,15,22)\}$

$B_6=\{(6,13,24),(6,14,22),(6,15,23)\}$

$B_7=\{(7,16,25),(7,17,26),(7,18,27)\}$

$B_8=\{(8,16,26),(8,17,27),(8,18,25)\}$

$B_9=\{(9,16,27),(9,17,25),(9,18,26)\}$

$B_{10}=\{(10,22,7),(10,23,8),(10,24,9)\}$

$B_{11}=\{(11,22,8),(11,23,9),(11,24,7)\}$

$B_{12}=\{(12,22,9),(12,23,7),(12,24,8)\}$

$B_{13}=\{(13,25,1),(13,26,2),(13,27,3)\}$

$B_{14}=\{(14,25,2),(14,26,3),(14,27,1)\}$

$B_{15}=\{(15,25,3),(15,26,1),(15,27,2)\}$

$B_{16}=\{(16,19,4),(16,20,5),(16,21,6)\}$

$B_{17}=\{(17,19,5),(17,20,6),(17,21,4)\}$

$B_{18}=\{(18,19,6),(18,20,4),(18,21,5)\}$

$B_{19}=\{(19,7,13),(19,8,14),(19,9,15)\}$

$B_{20}=\{(20,7,14),(20,8,15),(20,9,13)\}$

$B_{21}=\{(21,7,15),(21,8,13),(21,9,14)\}$

$B_{22}=\{(22,1,16),(22,2,17),(22,3,18)\}$

$B_{23}=\{(23,1,17),(23,2,18),(23,3,16)\}$
$B_{24}=\{(24,1,18),(24,2,16),(24,3,17)\}$
$B_{25}=\{(25,4,10),(25,5,11),(25,6,12)\}$
$B_{26}=\{(26,4,11),(26,5,12),(26,6,10)\}$
$B_{27}=\{(27,4,12),(27,5,10),(27,6,11)\}$.
Furthermore, construct 27 trefoils $\tilde{B}_i$ ($i=1,2,\ldots,27$) by reversing all directions of edges in B_i. Then they comprise a balanced trefoil decomposition of $K^*_{9,9,9}$.
Applying Theorem 7, $K^*_{n,n,n}$ has a balanced trefoil decomposition.

Therefore, we have the following theorem.

Main Theorem 2. $K^*_{n_1,n_2,n_3}$ has a balanced trefoil decomposition if and only if $n_1=n_2=n_3\equiv 0 \pmod 9$.

References

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