Tower Construction of Planar Coverings of Graphs

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Abstract

We shall show that a connected graph G_0 is projective-planar if and only if it has a series of double coverings $G_n \to G_{n-1} \to \cdots \to G_0$ with G_n planar, called a *planar tower*, and that a connected graph is projective-planar if and only if it has a projective-planar double covering. This works as an evidence supporting Negami's planar cover conjecture.

Introduction

Our graphs are simple and finite. A graph $\tilde{G}$ is called an *(n-fold) covering* of a graph G with a *projection* $p: \tilde{G} \to G$ if there is an n-to-one surjection $p: V(\tilde{G}) \to V(G)$ which sends the neighbors of each vertex $v \in V(\tilde{G})$ bijectively to those of $p(v)$. In particular, if there is a subgroup A in the automorphism group $\mathrm{Aut}(\tilde{G})$ such that $p(u)=p(v)$ whenever $\tau(u)=v$ for some $\tau \in A$, then $\tilde{G}$ is called a *regular covering*. It is easy to see that a 2-fold (or *double*) covering is necessarily a regular one.

A graph is said to be *projective-planar* if it can be embedded in the projective plane. The author [9] has discussed the relation between planar double coverings and embeddings of graphs in the projective plane, and established the following characterization of projective-planar graphs:

Theorem 1. (Negami [9]) *A connected graph is projective-planar if and only if it has a planar double covering.*

Furthermore, he has proved the following theorem, which extends Theorem 1 with "regular" instead of "double":

Theorem 2. (Negami [10]) *A connected graph is projective-planar if and only if it has a planar regular covering.*

These theorems motivated him to propose the following conjecture. This is called "the 1-2-∞ conjecture" or "Negami's planar cover conjecture":

Conjecture 1. (Negami [10], 1986) *A connected graph is projective-planar if and only if it has a planar covering.*

There have been many papers on studies around this conjecture, but the sufficiency is still open. At present, we have the following theorem, combining the results in [2, 3, 6, 8, 10, 11]:

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Theorem 3. (Archdeacon, Fellows, Hliněný and Negami) *If $K_{1,2,2,2}$ has no planar covering, then Conjecture 1 is true.*

In this paper, we shall give a partial result or an evidence supporting Conjecture 1, introducing a new notion on coverings, as follows.

Let $p_i : G_i \to G_{i-1}$ be a double covering projection from G_i to G_{i-1}. A series $G_n \to G_{n-1} \to \cdots \to G_0$ is called a *planar tower* of G_0 (of height n) if the top graph G_n is planar. The composition $p = p_1 p_2 \cdots p_n : G_n \to G_0$ of covering projections is a 2^n-fold covering projection from G_n to G_0 and is said to be obtained by *tower construction*.

Theorem 4. *A connected graph is projective-planar if and only if it has a planar tower.*

Since a planar covering obtained by tower construction is not regular in general, this theorem covers a part which Theorem 2 does not. We shall prove Theorem 4, as a consequence of the following theorem, which extends Theorem 1, relaxing the planarity of double coverings:

Theorem 5. *A connected graph is projective-planar if and only if it has a projective-planar double covering.*

Hliněný [7] has proposed the conjecture that a connected graph is projective-planar if and only if it has a projective-planar covering and has shown that it is equivalent to Conjecture 1. Theorem 5 gives us a partial answer to his conjecture.

1. Double coverings of $K_{1,2,2,2}$

As well as for Conjecture 1, we need to analyze the coverings of $K_{1,2,2,2}$. First we shall prepare the following lemma to decide their projective-planarity. Note that the subgraphs H_1 and H_2 discussed in the lemma are the same one as "disjoint k-subgraphs" defined by Archdeacon in [1].

Lemma 6. *Let G be a connected graph such that:*

(i) *There exist two disjoint subgraphs H_1 and H_2 of G each of which is isomorphic to either K_4 or $K_{2,3}$.*

(ii) *Each vertex of H_i is adjacent to a vertex in $G - V(H_j)$ for $\{i,j\} = \{1,2\}$.*

(iii) *Both $G - V(H_1)$ and $G - V(H_2)$ are connected.*

Then G is not projective-planar.

Proof. Suppose that G is embedded in the projective plane. Then there is a 2-cell region D^2 which contains one of H_1 and H_2 and is disjoint from the other. Assume that H_1 is contained in D^2. Since $G - V(H_1)$ is connected, we can suppose that D^2 is disjoint from $G - V(H_1)$. Then each edge joining H_1 and $G - V(H_1)$ must cross the boundary of D^2. This implies that H_1 is outerplanar, but this is impossible since neither K_4 nor $K_{2,3}$ is outerplanar. ■

The following lemma is a main part of our arguments to prove Theorems 4 and 5:

Lemma 7. *$K_{1,2,2,2}$ does not have any projective-planar double covering.*

Proof. The graph $K_{1,2,2,2}$ can be regarded as the join of a graph T isomorphic to $K_{2,2,2}$ with an extra vertex x. Let $\{a_1, a_2, b_1, b_2, c_1, c_2\}$ be the six vertices of T labeled so that two vertices are adjacent only when they have different alphabets. Then there are eight triangles $a_ib_jc_k$ $(i, j, k \in \{1, 2\})$ and T can be embedded on the sphere so that it forms the octahedron with faces $a_ib_jc_k$.

Consider subgraphs in $K_{1,2,2,2}$ isomorphic to either K_4 or $K_{2,3}$ and categorize them into the following three:

1. Each of the subgraphs induced by $\{a_i, b_j, c_k, x\}$ is isomorphic to K_4.

2. The union of any cycle uw_1vw_2 of length 4 in T with a path uxv forms a subgraph isomorphic to $K_{2,3}$.

3. The union of three paths $a_1b_1a_2$, $a_1b_2a_2$, $a_1c_ka_2$ forms a subgraph isomorphic to $K_{2,3}$. Any permutation over $\{a, b, c\}$ generates this type of a subgraph.

Let $p: \tilde{K} \rightarrow K_{1,2,2,2}$ be a double covering and let H be a subgraph in $K_{1,2,2,2}$ of one of the above three types. Suppose that H can be *lifted* to $\tilde{K}$, that is, $p^{-1}(H)$ consists of two components, say H_1 and H_2, and each of them is isomorphic to H. It is clear that Conditions (i) and (ii) in Lemma 6 hold for these H_1 and H_2.

First suppose that H is isomorphic to K_4 and let J be any component of $p^{-1}(T-V(H))$. If J is joined to only one of H_1 and H_2 with edges, say H_1, then J must be a cycle of length 3 obtained as a lift of $a_{3-i}b_{3-j}c_{3-k}$ and $J \cup H_1$ induces one component of $\tilde{K}$, isomorphic to $K_{1,2,2,2}$. Thus, $\tilde{K}$ is isomorphic to $K_{1,2,2,2} \cup K_{1,2,2,2}$ and is not projective-planar. Otherwise, all components of $p^{-1}(T-V(H))$ are joined to H_2 with edges and they form a connected subgraph $\tilde{K}-V(H_1)$ with H_2. Thus, Condition (iii) holds in this case and hence $\tilde{K}$ is not projective-planar by Lemma 6.

Suppose that H is of the second type. Similarly to the previous case, let J be any component of $p^{-1}(T-V(H))$ and suppose that J is joined to only H_1 with edges. First assume that H contains the cycle $C=a_1b_1a_2c_1$ as uw_1vw_2 and the path a_1xa_2 as uxv. Then J is a lift of an edge b_2c_2 and the both ends of J are adjacent to all lifts of a_1, a_2 and x in H_1. Let $\tilde{a}_1$ and $\tilde{x}$ be the lifts of a_1 and x in H_1, respectively. Then $J \cup \{\tilde{a}_1, \tilde{x}\}$ induces a subgraph in $\tilde{K}$ which projects isomorphically to a subgraph of the first type. Thus, we can assume that J is joined to H_2 in this case and hence $\tilde{K}-V(H_1)$ is connected. This implies that $\tilde{K}$ is not projective-planar by Lemma 6.

In the remaining cases with H of the second type, we can find those subgraphs already discussed in the previous cases, as follows. If H consists of the above C and the path b_1xc_1, then either the subgraph induced by H_1 contains a subgraph isomorphic to K_4, or there is a path in $\tilde{K}$ projecting to a_1xa_2. On the other hand, if H consists of the cycle $C'=a_1b_1a_2b_2$ and the path a_1xa_2, then J consists of a single vertex which projects to c_1 or c_2, say c_1, and the vertex is adjacent to all vertices of the lift of C'. In this case, there is a subgraph in $\tilde{K}$ which projects to $C \cup a_1xa_2$.

Finally suppose that H is of the third type. Then $p^{-1}(K_{1,2,2,2}-V(H))$ has two components and each of which consists only of an edge projecting to $c_{3-k}x$. If one of the components is joined only to H_1 with edges, then we can find a subgraph in $\tilde{K}$ isomorphic to K_4 and conclude that $\tilde{K}$ is not projective-planar, as well as in the previous case. Otherwise, Condition (iii) in Lemma 6 holds and $\tilde{K}$ is not projective-planar, again.

To complete the proof, it suffices to show that there is no projective-planar double covering of $K_{1,2,2,2}$ such that any subgraph isomorphic to K_4 or $K_{2,3}$ cannot be lifted. To describe a possible double covering, we use a *voltage assignment* to $E(K_{1,2,2,2})$ with $\mathbf{Z}_2=\{0, 1\}$. (See [5] for the theory of voltage graphs.) We may assume that each edge incident to x has voltage 0 since they form a spanning tree of $K_{1,2,2,2}$ and consider only the voltages over edges of T.

Those voltages must satisfy the following conditions to exclude the lift of subgraphs of corresponding types. The *voltage* of a path or a cycle is defined as the summation of the voltages along it. Any path can be lifted while a cycle can be lifted if and only if its voltage is 0.

1. At least one of three edges on the triangle $a_i b_j c_k$ has voltage 1 for $i, j, k \in \{1, 2\}$.

2. At least one of two paths of length 2 between any pair of vertices has voltage 1.

3. At least one of three paths of length 2 between any nonadjacent pair of vertices has voltage 0 and the others have voltage 1.

Let $\tilde{K}$ be a double covering of $K_{1,2,2,2}$ derived by a given voltage assignment with the above conditions and put $\tilde{T}=p^{-1}(T)$. Then $\tilde{T}$ has 12 vertices and 24 edges. By Euler's formula, $\tilde{T}$ must have 13 faces whenever it is embedded in the projective plane. Furthermore, we have $\sum_{i\geq 3} iF_i = 2\cdot 24 = 48$, where F_i stands for the number of faces of size i. Since $48/13 < 4$, then $\tilde{T}$ has a triangular face and hence there is a triangle in T, say $a_1 b_1 c_1$, with voltage 0.

By Condition 1, we may assume that $a_1 b_1$, $a_1 c_1$ and $b_1 c_1$ have voltage 1, 1 and 0, respectively. By Condition 1 for $a_2 b_1 c_1$ and Condition 2 for $\{a_1, a_2\}$, we may assume that $a_2 b_1$ and $a_2 c_1$ have voltage 0 and 1, respectively, up to symmetry. If $b_1 c_2$ had voltage 0, then $a_2 c_2$ would have voltage 1 by Condition 1 for $a_2 b_1 c_2$, but it would have voltage 0 by Condition 2 for $\{c_1, c_2\}$, a contradiction. Thus, $b_1 c_2$ must have voltage 1.

Suppose that $a_2 c_2$ has voltage 0. Condition 3 for $\{c_1, c_2\}$ forces $a_1 c_2$ to have voltage 1. In this case, the path $a_1 b_2 a_2$ can have neither voltage 0 nor 1 by Conditions 2 and 3 for $\{a_1, a_2\}$, a contradiction. Conversely, if $a_2 c_2$ has voltage 1, then the path $c_1 b_2 c_2$ can have neither voltage 0 nor 1 by Conditions 2 and 3 for $\{c_1, c_2\}$, a contradiction. Therefore, we cannot define a voltage assignment with Conditions 1, 2 and 3. ■

2. Proof of theorems

Proof of Theorem 5. The necessity follows from Theorem 1, so it suffices to show the sufficiency. Thus, we shall show that if a connected graph G is not projective-planar, then G does not have any projective-planar double covering.

A graph H is called a *minor* of a graph G if H can be obtained from G by contracting and deleting some edges. It is easy to see that if G has a projective-planar double covering, then so does H. Thus, it suffices to show that every minor-minimal graph among those graphs that are not projective-planar does not have a projective-planar double covering. Such minor-minimal graphs have been already identified in [1] and [4]; they are 35 in number and three of them are disconnected. We do not need those disconnected ones.

Let G_Y be a graph with a vertex v of degree 3 and let v_1, v_2 and v_3 be the three neighbors of v. A *Y-Δ transformation* is to add three new edges v_1v_2, v_2v_3 and v_3v_1 after deleting v. Let G_Δ be a graph obtained from G_Y by a Y-Δ transformation. It is easy to see that if G_Y has a projective-planar covering, then so does G_Δ. It has been known that the 32 minor-minimal graphs can be classified into 11 families, up to Y-Δ transformations, and that every member in 10 families not including $K_{1,2,2,2}$ does not have any planar covering.

Taking the universal covering of the projective plane, which is homeomorphic to the sphere, derives a planar double covering of any graph embedded in the projective plane, and hence if G has a projective-planar double covering, then G has a planar 4-fold covering. Thus, it suffices to show that any graph which can be deformed into $K_{1,2,2,2}$ by Y-Δ transformations does not have a projective-planar double covering. It is however clear since $K_{1,2,2,2}$ itself does not by Lemma 7. This completes the proof. ■

Note that our arguments in the previous proof work for a proof of Theorem 3 if we replace "projective-planar double coverings" with "planar coverings".

Proof of Theorem 4. We shall show only the sufficiency, using induction on the height n of a planar tower $G_n \to G_{n-1} \to \cdots \to G_1 \to G_0$. If $n=1$, then G_0 is projective-planar, by Theorem 1. If $n>1$, then $G_n \to G_{n-1} \to \cdots \to G_1$ is a planar tower of G_1 of height $n-1$ and hence G_1 is projective-planar, by the induction hypothesis. Since $p_1 : G_1 \to G_0$ is a projective-planar double covering, G_0 is projective-planar, by Theorem 5. This completes the induction. ■

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