Circuits of Antimatoriods and Dilworth's Decomposition Theorem

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Abstract

Antimatroids are antipodal to matroids, which are considered as combinatorial abstraction of convexity. We consider an extension of Dilworth's decomposition theorem for partially ordered sets, which states that the maximum size of antichains is equal to the minimum number of chains which cover the ground set for any partially ordered set. In particular, we investigate the relationship between the extended statement and circuits of antimatroids.

1 Antimatroids

In a paper of 1950, Dilworth[1] showed one of the most famous theorems for partially ordered sets, or posets. That is, for any poset, the maximum size of antichains is equal to the minimum number of chains which cover the ground set. In this paper, we consider an extension of this theorem to antimatroids. Antimatroids are known as a combinatorial abstraction of the concept of "convexity," which are "antipodal" to matroids in many respects. For example, matroids can be defined by a closure operator which satisfies the exchange axiom, while antimatroids can be defined by a closure operator which satisfies the anti-exchange axiom. See [2, 4, 5, 6, 7, 8, 9] for details of antimatroids, greedoids, which includes antimatroids and matroids as subclasses, and convex geometries, which is known as dual of antimatroids.

Let E be a non-empty finite set. A family $\mathcal{F}$ of subsets of E is an *antimatroid* on E if it satisfies the following conditions:

(1) $\emptyset \in \mathcal{F}$, $E \in \mathcal{F}$
(2) $X \in \mathcal{F} \setminus \{\emptyset\}$ implies $X \setminus \{e\} \in \mathcal{F}$ for some $e \in X$,
(3) $X, Y \in \mathcal{F}$ implies $X \cup Y \in \mathcal{F}$

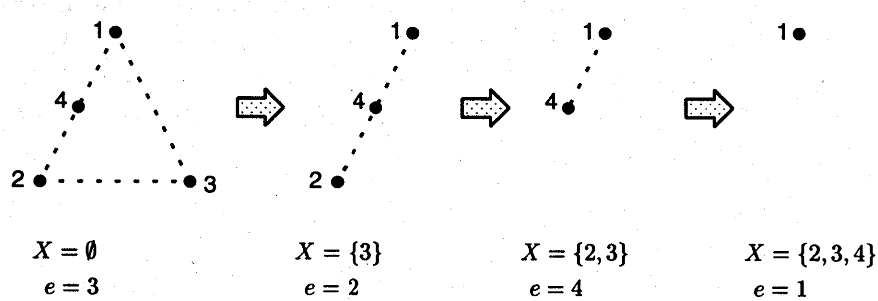
Members of an antimatroid are called *feasible sets*, and E is called the *ground set* of $\mathcal{F}$.

We can obtain many classes of antimatroids by shelling processes or searching processes. By *shelling* processes, we mean repeated elimination of suitable elements until all the elements are removed. We see some examples of shelling processes.

Example 1.1 (convex shelling of points on the Euclidean space). Let $E \subseteq \mathbb{R}^n$ be a finite set of points on the Euclidean space. Then, we consider the following procedure:

1. Set $X := \emptyset$ and $C := \{\emptyset\}$
2. while $X \neq E$ repeat :
   2.1 Cloose a vertex of the convex hull of $E \setminus X$, say e.
   2.2 Reset $X := X \cup \{e\}$ and $C := C \cup \{X \cup \{e\}\}$.
   2.3 Return to the head of this repetition.

Here, the *convex hull* of X is the minimal closed set which includes all points of X. For example, we consider the case of the figure below.



This figure indicates how the procedure executes, and now by this procedure we have $C = \{\emptyset, \{3\}, \{2, 3\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}$. The family C depends on the order to choose e in each iteration. Here, we enumerate all of the possible cases:

$$
\begin{aligned}
C_1 &= \{\emptyset, \{1\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 3, 4\}\},\\
C_2 &= \{\emptyset, \{1\}, \{1, 2\}, \{1, 2, 4\}, \{1, 2, 3, 4\}\},\\
C_3 &= \{\emptyset, \{1\}, \{1, 3\}, \{1, 2, 3\}, \{1, 2, 3, 4\}\},\\
C_4 &= \{\emptyset, \{1\}, \{1, 3\}, \{1, 3, 4\}, \{1, 2, 3, 4\}\},\\
C_5 &= \{\emptyset, \{1\}, \{1, 4\}, \{1, 3, 4\}, \{1, 2, 3, 4\}\},\\
C_6 &= \{\emptyset, \{1\}, \{1, 4\}, \{1, 2, 4\}, \{1, 2, 3, 4\}\},\\
C_7 &= \{\emptyset, \{2\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 3, 4\}\},\\
C_8 &= \{\emptyset, \{2\}, \{1, 2\}, \{1, 2, 4\}, \{1, 2, 3, 4\}\},\\
C_9 &= \{\emptyset, \{2\}, \{2, 3\}, \{1, 2, 3\}, \{1, 2, 3, 4\}\},\\
C_{10} &= \{\emptyset, \{2\}, \{2, 3\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\},\\
C_{11} &= \{\emptyset, \{2\}, \{2, 4\}, \{1, 2, 4\}, \{1, 2, 3, 4\}\},\\
C_{12} &= \{\emptyset, \{2\}, \{2, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\},\\
C_{13} &= \{\emptyset, \{3\}, \{1, 3\}, \{1, 2, 3\}, \{1, 2, 3, 4\}\},
\end{aligned}
$$

$C_{14} = \{\emptyset, \{3\}, \{1, 3\}, \{1, 3, 4\}, \{1, 2, 3, 4\}\}$,
$C_{15} = \{\emptyset, \{3\}, \{2, 3\}, \{1, 2, 3\}, \{1, 2, 3, 4\}\}$,
$C_{16} = \{\emptyset, \{3\}, \{2, 3\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}$.

Then, we define $\mathcal{F}$ as

$$\mathcal{F} = \bigcup_{i=1}^{16} C_i,$$
$$= \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}$$

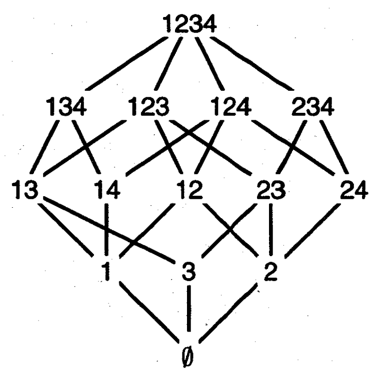
We can easily check that $\mathcal{F}$ is an antimatroid, which is called the *convex shelling* of E.

In other words, we can represent this antimatroid $\mathcal{F}$ as:

(4) $\mathcal{F} = \{X \subseteq E: E \setminus X = \text{conv.hull}(E \setminus X) \cap E\}$,

where conv.hull($E \setminus X$) is the convex hull of $E \setminus X$.

Often, we represent an antimatroid by the Hasse diagram as the following figure.

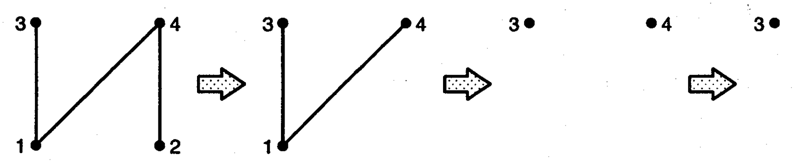


Now, we see another example of shelling processes.

Example 1.2 (poset shelling). Let $(E, \leq)$ be a poset. Then we consider the following procedure:

1. Set $X := \emptyset$ and $C := \{\emptyset\}$.
2. While $X \neq E$ repeat:
   2. 1 Choose a minimal element of $E \setminus X$, say e.
   2. 2 Reset $X := X \cup \{e\}$ and $C := C \cup \{X \cup \{e\}\}$.
   2. 3 Return to the head of this repetition.

For example, we consider the case of the figure below.



$X = \emptyset$, $e = 2$ $\quad$ $X = \{2\}$, $e = 1$ $\quad$ $X = \{1,2\}$, $e = 4$ $\quad$ $X = \{1,2,4\}$, $e = 3$

This figure indicates how the procedure executes, and now by this procedure we have $C = \{\emptyset, \{2\}, \{1, 2\}, \{1, 2, 4\}, \{1, 2, 3, 4\}\}$. The family C depends on the order to choose e in each iteration.
Then,

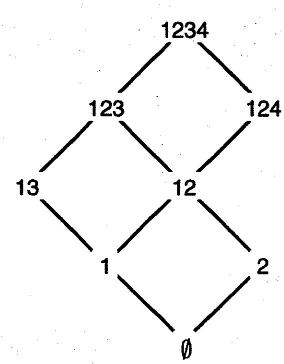
$$\mathcal{F} = \bigcup C$$
$$= \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 3, 4\}\}$$

forms an antimatroid, which is called the *poset shelling* of $(E, \leq)$.

An *order ideal* of the poset $(E, \leq)$ is a subset $X \subseteq E$ such that $b \in X$, $a \leq b$ imply $a \in X$. We also define a poset shelling $\mathcal{F}$ by order ideals as follows:

(5) $\mathcal{F} = \{X \subseteq E$: X is an order ideal of $(E, \leq)\}$.

The following figure shows the representation by the Hasse diagram of this antimatroid.



The only difference of the procedures in Example 1.1 and Example 1.2 is Step 2.1. In Example 1.1, Step 2.1 of the procedure lets us choose a vertex of the convex hull, while in Example 1.2, Step 2.1 of the procedure lets us choose a minimal element. Thus, by modifying Step 2.1, we can generate some antimatroids in similar ways.

Example 1.3 (vertex shelling of a tree / edge shelling of a tree). Let $G = (V, E)$ be a tree. A *leaf* of G is a vertex with degree 1. If we delete leaves step by step, then we obtain an antimatroid on V, which is called the *vertex shelling* of the tree G. Analogously, we define the *edge shelling* of a tree.

Example 1.4 (double shelling of a poset). Let $(E, \leq)$ be a poset. Now, we consider deleting minimal elements or maximal elements repeatedly. Then, we obtain an antimatroid on E, which is called the *double shelling* of the poset $(E, \leq)$.

Example 1.5 (simplicial shelling of a triangulated graph). A graph $G=(V, E)$ is *triangulated* (or chordal) if it contains no induced cycles other than triangles. A vertex of a graph is *simplicial* if its neighbors form a complete subgraph.

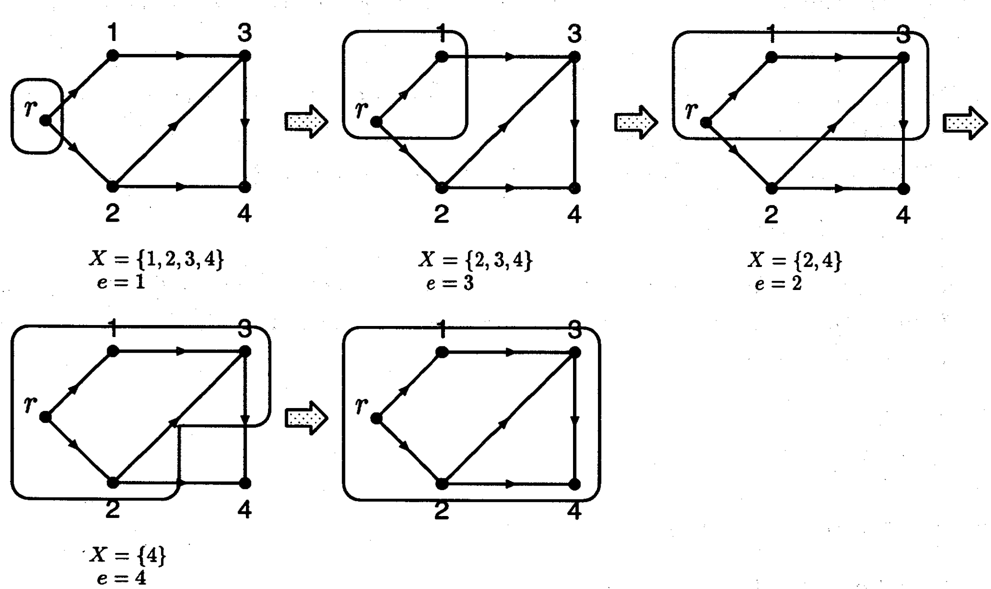
Let $G=(V, E)$ be a triangulated graph. If we eliminate a simplicial vertex in each iteration, then we obtain an antimatroid on V, which is called the *simplicial shelling* of the triangulated graph G.

A *search* is another process to obtain antimatroids, which means a gathering process of adjacent elements. Let us see an example.

Example 1.6 (point search of a rooted directed graph). Let $G=(V \cup \{r\}, E)$ be a directed graph with a specified vertex r called the *root*. Then we consider the following procedure:

1. Set $X := \emptyset$ and $C := \{\emptyset\}$.
2. While $X \neq V$ repeat:
   2.1 Choose a vertex of $V \setminus X$ adjacent to a vertex of $X \cup \{r\}$, say e.
   2.2 Reset $X := X \cup \{e\}$ and $C := C \cup \{X \cup \{e\}\}$
   2.3 Return to the head of this repetition.

For example, we consider a rooted graph G with the vertices $\{r, 1, 2, 3, 4\}$ and edges $\{(r, 1), (r, 2), (1, 3), (2, 3), (2, 4), (3, 4)\}$, and the search process as follows:



By this procedure we have $C=\{\emptyset, \{1\}, \{1, 3\}, \{1, 2, 3\}, \{1, 2, 3, 4\}\}$. The family C depends on the order that we choose e in each iteration. Then,

$$\begin{aligned}\mathcal{F} &= \bigcup C \\ &= \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \\ &\quad \{1, 2, 3, 4\}\}\end{aligned}$$

forms an antimatroid on V, which is called the *point search* of the rooted directed graph G.

Analogously, we also consider point searches of rooted undirected graphs, line searches of rooted directed graphs, and line searches of rooted undirected graphs.

It is known that an antimatroid forms a *semimodular lattice* with respect to set-inclusion. See [10] for comprehensive results on semimodular lattices.

2 Dilworth-type antimatroids

Let us recall Dilworth's decomposition theorem.

Let $P=(E, \leq)$ be a poset. A subset $X \subseteq E$ is a *chain* of P if any two distinct elements of X are comparable. Similarly, a subset $X \subseteq E$ is an *antichain* of P if any two distinct elements of X are incomparable. We denote the family of the chains of P by $C(P)$, and the family of the antichains of P by $\mathcal{A}(P)$.

Proposition 2.1 (Dilworth's decomposition theorem [1]). *Let $P = (E, \leq)$ be a poset. Then,*

$$\max\{|A| : A \in \mathcal{A}(P)\} = \min\left\{t : C_1, C_2, \ldots, C_t \in C(P), \bigcup_{i=1}^{t} C_i = E\right\}, \tag{6}$$

that is, the maximum size of antichains of P is equal to the minimum number of chains of P which cover E.

Now, we consider an extension of Dilworth's decomposition theorem to antimatroids. First, we have to consider antimatroidal analogues of chains and antichains.

Let $\mathcal{F}$ be an antimatroid on E, and $A \subseteq E$. The *trace* on A is defined by

$$\mathcal{F} : A = \{A \cap X : X \in \mathcal{F}\}, \tag{7}$$

which means that, if we restrict E to A and consider shelling processes on A in the same way as we obtain $\mathcal{F}$ on E, then we obtain $\mathcal{F} : A$. Notice that the trace of an antimatroid is also an antimatroid on A. A subset $A \subseteq E$ is *free* if

$$\mathcal{F} : A = 2^A, \tag{8}$$

while A is *linear* if

$$\mathcal{F} : A = \{\emptyset\} \cup \{\{e_1, \ldots, e_i\} : i = 1, \ldots, k\} \tag{9}$$

for some numbering $A = \{e_1, \ldots, e_k\}$.

Remark 2.2. *Let $\mathcal{F}$ be the poset shelling of poset $P=(E, \leq)$, and $A\subseteq E$. A is a free set of $\mathcal{F}$ if and only if it is an antichain of P, and A is a linear set of $\mathcal{F}$ if and only if it is a chain of P.*

For an antimatroid $\mathcal{F}$, we denote the family of the free sets by Free($\mathcal{F}$), and the family of the linear sets by Lin($\mathcal{F}$). Traces and free sets were first introduced in [8], while linear sets did not appear in the past literature. We investigate some properties of free sets and linear sets.

Lemma 2.3. *Let $\mathcal{F}$ be an antimatroid on E.* Free($\mathcal{F}$) *and* Lin($\mathcal{F}$) *are hereditary, that is,*

(10) $X\in \mathrm{Free}(\mathcal{F})$ *and* $X'\subseteq X$ *imply* $X'\in \mathrm{Free}(\mathcal{F})$,

(11) $X\in \mathrm{Lin}(\mathcal{F})$ *and* $X'\subseteq X$ *imply* $X'\in \mathrm{Lin}(\mathcal{F})$.

Proof. Let $X\in \mathrm{Free}(\mathcal{F})$ and $X'\subseteq X$. Then,

$$\{Y\cap X' : Y\in\mathcal{F}\} = \{(Y\cap X)\cap X' : Y\in\mathcal{F}\} = \{Z\cap X' : Z\in\mathcal{F} : X\}.$$

Therefore, $\mathcal{F} : X' = (\mathcal{F} : X) : X'$. Since $2^E : X = \{Y\cap X : Y\in 2^E\} = 2^X$ for any $A\subseteq E$, we have $\mathcal{F} : X'=(\mathcal{F} : X) : X'= 2^X : X'= 2^{X'}$ for $X\in \mathrm{Free}(\mathcal{F})$. Hence $X'\in \mathrm{Free}(\mathcal{F})$. We have a similar discussion on Lin($\mathcal{F}$). □

Lemma 2.4. *Let $\mathcal{F}$ be an antimatroid on E. Then, for any $X\in \mathrm{Free}(\mathcal{F})$ and any $Y\in \mathrm{Lin}(\mathcal{F})$, we have* $|X\cap Y|\leq 1$.

Proof. By Lemma 2.3, we have $X\cap Y\in \mathrm{Free}(\mathcal{F})$. Similarly, $X\cap Y\in \mathrm{Lin}(\mathcal{F})$. Therefore, $\mathcal{F} : X\cap Y=2^{X\cap Y} =\{\emptyset\}\cup\{\{e_1, \ldots, e_i\} : i=1, \ldots, k\}$ for $X\cap Y = \{e_1, \ldots, e_k\}$. Hence, $|X\cap Y|$ must be at most one. □

Lemma 2.5. *Let $\mathcal{F}$ be an antimatroid on E, and $A\subseteq E$. Then,*

(12) $|A|\leq 1 \iff A\in \mathrm{Lin}(\mathcal{F})$ *and* $A\in \mathrm{Free}(\mathcal{F})$,

(13) $|A| = 2 \Rightarrow$ *either* $A\in \mathrm{Lin}(\mathcal{F})$ *or* $A\in \mathrm{Free}(\mathcal{F})$.

Proof. It is easily checked from the definitions of free sets and linear sets. □

Lemma 2.6. *Let $\mathcal{F}$ be an antimatroid on E. Then,*

(14) $\max\{|X| : X\in \mathrm{Free}(\mathcal{F})\}\leq \min\left\{t : Y_1, \cdots, Y_t\in \mathrm{Lin}(\mathcal{F}),\ \bigcup_{i=1}^{t} Y_i=E\right\}$.

Proof. Without loss of generality, we assume that $Y_1, \ldots, Y_t\in \mathrm{Lin}(\mathcal{F})$ are all disjoint by Lemma 2.3. For any $X\in \mathrm{Free}(\mathcal{F})$ and for any disjoint $Y_1, \ldots, Y_t\in \mathrm{Lin}(\mathcal{F})$ such that $\bigcup Y_i=E$,

$$|X|\leq \sum_{i=1}^{t} |X\cap Y_i|\leq \sum_{i=1}^{t} 1=t,$$

using Lemma 2.4. □

We now introduce Dilworth-type antimatroids. Let $\mathcal{F}$ be an antimatroid on E. Free($\mathcal{F}$) be the family of the free sets of $\mathcal{F}$, and Lin($\mathcal{F}$) be the family of the linear sets of $\mathcal{F}$. The antimatroid $\mathcal{F}$ is *Dilworth-type* if it satisfies the following equality:

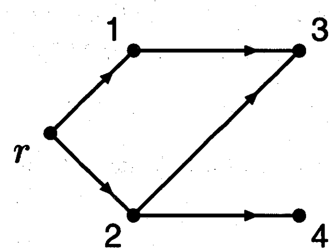
(15) $\max\{|X| : X \in \mathrm{Free}(\mathcal{F})\} = \min\left\{t : Y_1, \ldots, Y_t \in \mathrm{Lin}(\mathcal{F}), \bigcup_{i=1}^{t} Y_i = E\right\}.$

Poset shellings (Example 1.2) are Dilworth-type by Proposition 2.1 and Remark 2.2. However, there are some antimatroids which are not Dilworth-type. Here, we observe the point searches (Example 1.6) of two rooted directed graphs, one of which is Dilworth-type while the other is not.

Example 2.1. Let G be a directed graph with the vertices $\{r, 1, 2, 3, 4\}$ and the edges $\{(r, 1), (r, 2), (1, 3), (2, 3), (2, 4)\}$, and $\mathcal{F}$ be the point search of G. Then,

(16) $\mathrm{Lin}(\mathcal{F}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{2, 4\}\}$

(17) $\mathrm{Free}(\mathcal{F}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{3, 4\}, \{1, 3, 4\}\}$.

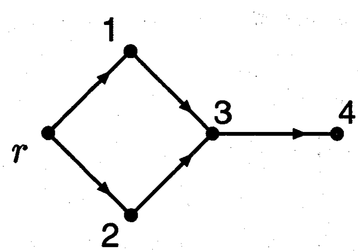


We can check that $\mathcal{F}$ is Dilworth-type.

Example 2.2. Let G be a directed graph with the vertices $\{r, 1, 2, 3, 4\}$ and the edges $\{(r, 1), (r, 2), (1, 3), (2, 3), (3, 4)\}$, and $\mathcal{F}$ be the point search of G. Then,

(18) $\mathrm{Lin}(\mathcal{F}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{3, 4\}\}$,

(19) $\mathrm{Free}(\mathcal{F}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}\}$.



We can check that $\mathcal{F}$ is not Dilworth-type.

In the next section, we introduce circuits of antimatoids and show that an antimatroid is not Dilworth-type if all of its circuits have cardinality more than two.

3 Circuits of antimatroids and Dilworth-type antimatroids

First, we introduce the concept of circuits of antimatroids. Let $\mathcal{F}$ be an antimatroid on E. A subset $C \subseteq E$ is a *circuit* of $\mathcal{F}$ if C is minimal non-free set, that is, C is not free and every proper subset of C is free. Let us see some example. If $\mathcal{F}$ is a convex shelling on the 2-dimensional Euclidean plane (Example 1.1), then the circuits are triples of colinear points and quadruples in which one point is in the interior of the triangle spanned by the other three. If $\mathcal{F}$ is a shelling of a poset $P=(E, \leq)$ (Example 1.2), then the circuits are pairs $\{a, b\}$ such that $a<b$. If $\mathcal{F}$ is a vertex shelling of a tree $G=(V, E)$ (Example 1.3), then the circuits are triples of vertices lying on a path. If $\mathcal{F}$ is a double shelling of a poset $P=(E, \leq)$ (Example 1.4), then the circuits are triples $\{a, b, c\}$ such that $a<b<c$. Note that we have the characterization of antimatroids via circuits [3] analogous to the case of matroids. However, it is not a topic for this paper.

Let C be a circuit of an antimatroid $\mathcal{F}$ on E. Then, it is known that there exists a unique element $r \in C$ such that $\mathcal{F}:C=2^C \setminus \{r\}$ (see [8, 9], etc.). We call the element r the *root* of the circuit C. Sometimes we denote a circuit C with its root r by (C, r) in order to specify the root of C.

Lemma 3.1. *Let $\mathcal{F}$ be an antimatroid on E and $C \subseteq E$ be a circuit of $\mathcal{F}$. Then $|C|=2$ if and only if $C \in \mathrm{Lin}(\mathcal{F})$.*

Proof. Set $C=\{r, e\}$ with the root r. From the definition of the root, $\mathcal{F}:\{r, e\}=\{\emptyset, \{e\}, \{r, e\}\}$. Therefore , $\{r, e\}$ is linear. The converse can be similarly checked. □

We see an interesting fact in [8]. An antimatroid $\mathcal{F}$ on E is *Boolean* if $\mathcal{F}=2^E$. Otherwise, it is called non-Boolean.

Proposition 3.2 (Characterization of poset shelings via circuits [8]). *Let $\mathcal{F}$ be a non-Boolean antimatroid on E. Then, $\mathcal{F}$ is a poset shelling if and only if every circuit of $\mathcal{F}$ has cardinality two.*

From Proposition 3.2 and Remark 2.2, we can see that, if every circuit of $\mathcal{F}$ has cardinality two, then $\mathcal{F}$ is Dilworth-type.

Now we characterize the antimatroids every circuit of which has cardinality more than two, and show that they are not Dilworth-type. An antimatroid $\mathcal{F}$ on E is *coatomic* if $E \setminus \{e\} \in \mathcal{F}$ for all $e \in E$, which means that we can leave any element as the last in the shelling process for a coatomic antimatroid. There are many coatomic antimatroids. For example, convex shellings of points on the Euclidean space (Example 1.1), vertex/edge shellings of trees (Example 1.3), double shellings of posets (Example 1.4), simplicial shellings of triangulated graphs (Example 1.5) are coatomic.

Lemma 3.3 (Characterization of coatomic antimatroids via circuits). *Let $\mathcal{F}$ be a non-Boolean antimatoid on E. Then, $\mathcal{F}$ is a coatomic antimatroid if and only if every circuit of $\mathcal{F}$ has cardinality more than two.*

Proof. Let $\mathcal{F}$ be coatomic, that is, $E \setminus \{e\} \in \mathcal{F}$ for any $e \in E$. Suppose that $\{a, b\}$ is a circuit of $\mathcal{F}$.

From Lemma 3.1, $\{a, b\}$ is linear. Therefore, $\mathcal{F}:\{a, b\}=\{\emptyset, \{a\}, \{a, b\}\}$. This implies $E\setminus\{a\}\notin\mathcal{F}$, which is a contradiction.

Conversely, let $\mathcal{F}$ be not coatomic, that is, $E\setminus\{e\}\notin\mathcal{F}$ for some $e\in E$. Here, we only have to show that $\{a, e\}$ is a ciruit for some $a\in E$, that is, $\{a, e\}$ is linear (Lemma 3.1.) Suppose that $\{a, e\}$ is free for any $a\in E\setminus\{e\}$. Then, for each $a\in E\setminus\{e\}$, there exists $A\in\mathcal{F}$ such that $e\notin A$ and $a\in A$. This implies that $E\setminus\{e\}=\bigcup\{A\in\mathcal{F}: e\notin A\}\in\mathcal{F}$, which is a contradiction. □

Lemma 3.4. *Let $\mathcal{F}$ be a coatomic antimatroid on E and $A\subseteq E$. Then $A\in\mathrm{Lin}(\mathcal{F})$ if and only if* $|A|\leq 1$.

Proof. It is checked directly from Lemma 2.5 and 3.3.

Theorem 3.5. *Any non-Boolean coatomic antimatroid is not Dilworth-type.*

Proof. Let $\mathcal{F}$ be anon-Boolean coatomic antimatroid on *E*. From Lemma 3.4,

$$\min\left\{t: Y_1, \ldots, Y_t\in\mathrm{Lin}(\mathcal{F}), \bigcup_{i=1}^{t} Y_i=E\right\}=|E|.$$

On the other hand, suppose that $\max\{|X|: X\in\mathrm{Free}(\mathcal{F})\}=|E|$. Then $\mathcal{F}$ must be Boolean, which is a contradiction. □

4 Summary

We summarize the results in the following table.

Size of circuits	Dilworth-type	Examples
all two	yes	poset shellings
all more than two	no	convex shellings of points on $\mathbb{R}^n$, shellings of trees, double shellings of posets, simplicial shellings of triangulated graphs, etc.
otherwise	yes/no	point searches of rooted digraphs, etc.

We consider point searches of rooted digraphs (Example 1.6) such that some circuits of them have size two and some of them have size more than two. In this class we have both Dilworth-type antimatroids and not Dilworth-type ones. Indeed, Example 2.1 is Dilworth-type, and the family of its circuits is $\{\{1, 2, 3\}, \{2, 4\}\}$. On the other hand, Example 2.2 is not Dilworth-type, and the family of its circuits is $\{\{1, 2, 3\}, \{1, 2, 4\}, \{3, 4\}\}$.

We finally remark that we have not yet known a characterization of Dilworth-type antimatroids and any algorithmic properties of them.

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