Isomorphisms of Some Cyclic Abelian Covers of Symmetric Digraphs II

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Abstract

Let D be a connected symmetric digraph, $\mathbf{Z}_p$ a cyclic group of prime order $p(>3)$ and Γ a group of automorphisms of D. We enumerate the number of Γ-isomorphism classes of g-cycic $\mathbf{Z}_p^3$-covers of D for any nonunit $g \in \mathbf{Z}_p^3$, where $\mathbf{Z}_p^3$ is the direct sum of three $\mathbf{Z}_p$.

1. Introduction

Graphs and digraphs treated here are finite and simple.

Let D be a symmetric digraph and A a finite group. A function $\alpha : A(D) \longrightarrow A$ is called alternating if $\alpha(y,x) = \alpha(x,y)^{-1}$ for each $(x,y) \in A(D)$. For $g \in A$, a g-cyclic A-cover (or g-cyclic cover) $D_g(\alpha)$ of D is the digraph as follows:

$V(D_g(\alpha)) = V(D) \times A$, and $((u,h),(v,k)) \in A(D_g(\alpha))$ if and only if $(u,v) \in A(D)$ and $k^{-1}h\alpha(u,v) = g$.

The natural projection $\pi : D_g(\alpha) \longrightarrow D$ is a function from $V(D_g(\alpha))$ onto $V(D)$ which erases the second coordinates. A digraph D' is called a cyclic A-cover of D if D' is a g-cyclic A-cover of D for some $g \in A$. In the case that A is abelian, then $D_g(\alpha)$ is simply called a cyclic abelian cover.

Let α and β be two alternating functions from $A(D)$ into A, and let Γ be a subgroup of the automorphism group $Aut\ D$ of D, denoted $\Gamma \leq Aut\ D$. Let $g,h \in A$. Then two cyclic A-covers $D_g(\alpha)$ and $D_h(\beta)$ are called Γ-isomorphic, denoted $D_g(\alpha) \cong_\Gamma D_h(\beta)$, if there exist an isomorphism $\Phi : D_g(\alpha) \longrightarrow D_h(\beta)$ and a $\gamma \in \Gamma$ such that $\pi\Phi = \gamma\pi$, i.e., the diagram

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$$\begin{array}{ccc} D_g(\alpha) & \xrightarrow{\Phi} & D_h(\beta) \\ \pi \downarrow & & \downarrow \pi \\ D & \xrightarrow{\gamma} & D \end{array}$$

commutes. Let $I = \{1\}$ be the trivial subgroup of automorphisms.

Cheng and Wells [1] discussed isomorphism classes of cyclic triple covers (1-cyclic $\mathbf{Z}_3$-covers) of a complete symmetric digraph. Mizuno and Sato [16,18] enumerated the number of I-isomorphism classes of g-cyclic $\mathbf{Z}_p^n$-covers and g-cyclic $\mathbf{Z}_{p^n}$-covers, and Γ-isomorphism classes of g-cyclic $\mathbf{Z}_p$-covers of a connected symmetric digraph D for any prime $p(>2)$. Furthermore, Mizuno, Lee and Sato [15] gave a formula for the the number of I-isomorphism classes of connected g-cyclic $\mathbf{Z}_p^n$-covers and connected g-cyclic $\mathbf{Z}_{p^n}$-covers of D for any prime $p(>2)$. Mizuno and Sato [16] gave a formula for the enumeration of Γ-isomorphism classes of g-cyclic $\mathbf{Z}_p \times \mathbf{Z}_p$-covers of D for any prime $p(>2)$.

Let G be a graph and A a finite group. Let $D(G)$ be the arc set of the symmetric digraph corresponding to G. Then a mapping $\alpha : D(G) \longrightarrow A$ is called an ordinary voltage assignment if $\alpha(v,u) = \alpha(u,v)^{-1}$ for each $(u,v) \in D(G)$. The (ordinary) derived graph G^α derived from an ordinary voltage assignment α is defined as follows:

$V(G^\alpha) = V(G) \times A$, and $((u,h),(v,k)) \in D(G^\alpha)$ if and only if $(u,v) \in D(G)$ and $k = h\alpha(u,v)$.

The graph G^α is called an A-covering of G. The A-covering G^α is an $|A|$-fold regular covering of G. Every regular covering of G is an A-covering of G for some group A (see [3]). Furthermore the 1-cyclic A-cover $D_1(\alpha)$ of a symmetric digraph D can be considered as the A-covering $\tilde{D}^\alpha$ of the underlying graph $\tilde{D}$ of D.

A general theory of graph coverings is developed in [4]. $\mathbf{Z}_2$-coverings (double coverings) of graphs were dealed in [5] and [22]. Hofmeister [6] and, independently, Kwak and Lee [12] enumerated the I-isomorphism classes of n-fold coverings of a graph, for any $n \in \mathbf{N}$. Dresbach [2] obtained a formula for the number of strong isomorphism classes of regular coverings of graphs with voltages in finite fields. The I-isomorphism classes of regular coverings of graphs with voltages in finite dimensional vector spaces over finite fields were enumerated by Hofmeister [7]. Hong, Kwak and Lee [9] gave the number of I-isomorphism classes of $\mathbf{Z}_n$-coverings, $\mathbf{Z}_p \bigoplus \mathbf{Z}_p$-coverings and D_n-coverings, n:odd, of graphs, respectively. Mizuno and Sato [19,20,21] presented the numbers of Γ-isomorphism classes of $\mathbf{Z}_p^n$-coverings of graphs for $n = 1,2,3$ and any prime $p(>2)$.

In the case of connected coverings, Kwak and Lee [14] enumerated the I-isomorphism classes of connected n-fold coverings of a graph G. Furthermore, Kwak, Chun and Lee [13] gave some formulas for the number of I-isomorphism classes of connected A-coverings of G when A is a finite abelian group or D_n.

We present the number of Γ-isomorphism classes of g-cyclic $\mathbf{Z}_p^3$-covers of connected symmetric digraphs for any element $g \neq 0 \in \mathbf{Z}_p^3$, where 0 is the unit of $\mathbf{Z}_p^3$ and any prime $p(>3)$.

2. Isomorphisms of cyclic $\mathbf{Z}_p^3$-covers

Let D be a connected symmetric digraph and A a finite abelian group. The group Γ of automorphisms of D acts on the set $C(D)$ of alternating functions from $A(D)$ into A as follows:

$$\alpha^{\gamma}(x,y) = \alpha(\gamma(x), \gamma(y)) \text{ for all } (x,y) \in A(D),$$

where $\alpha \in C(D)$ and $\gamma \in \Gamma$.

Let G be the underlying graph of D. The set of ordinary voltage assignments of G with voltages in A is denoted by $C^1(G;A)$. Note that $C(D) = C^1(G;A)$. Furthremore, let $C^0(G;A)$ be the set of functions from $V(G)$ into A. We consider $C^0(G;A)$ and $C^1(G;A)$ as additive groups. The homomorphism $\delta : C^0(G;A) \longrightarrow C^1(G;A)$ is defined by $(\delta s)(x,y) = s(x) - s(y)$ for $s \in C^0(G;A)$ and $(x,y) \in A(D)$. The 1-cohomology group $H^1(G;A)$ with coefficients in A is defined by $H^1(G;A) = C^1(G;A)/Im\,\delta$. For each $\alpha \in C^1(G;A)$, let $[\alpha]$ be the element of $H^1(G;A)$ which contains α.

The automorphism group $Aut\ A$ acts on $C^0(G;A)$ and $C^1(G;A)$ as follows:

$$(\sigma s)(x) = \sigma(s(x)) \text{ for } x \in V(D),$$

$$(\sigma\alpha)(x,y) = \sigma(\alpha(x,y)) \text{ for } (x,y) \in A(D),$$

where $s \in C^0(G;A)$, $\alpha \in C^1(G;A)$ and $\sigma \in Aut\ A$. A finite group B is said to have the isomorphism extension property(IEP), if every isomorphism between any two isomorphic subgroups $\mathcal{E}_1$ and $\mathcal{E}_2$ of B can be extended to an automorphism of B (see [9]). For example, the cyclic group $\mathbf{Z}_n$ for $n \in \mathbf{N}$, the dihedral group $\mathbf{D}_n$ for odd $n \geq 3$, and the direct sum of m copies of $\mathbf{Z}_p$(p: prime) have the IEP.

Mizuno and Sato [18] gave a characterization for two cyclic A-covers of D to be Γ-isomorphic.

Theorem 1 (18, Corollary 3) *Let D be a connected symmetric digraph, G the underlying graph of D, A a finite abelian group with the IEP, $g \in A$, $\alpha, \beta \in C(D)$ and $\Gamma \leq Aut\ D$. Assume that the order of g is odd. Then the following are equivalent:*

1. $D_g(\alpha) \cong_{\Gamma} D_g(\beta)$.
2. *There exist $\gamma \in \Gamma$, $\sigma \in Aut\ A$ and $s \in C^0(G;A)$ such that*

$$\beta = \sigma\alpha^{\gamma} + \delta s \text{ and } \sigma(g) = g.$$

Let $Iso(D,A,g,\Gamma)$ denote the number of Γ-isomorphism classes of g-cyclic A-covers of D. The following result holds.

Theorem 2 (18, Theorem 3) *Let D be a connected symmetric digraph, G the underlying graph of D, A a finite abelian group with the IEP, $g, h \in A$ and $\Gamma \leq Aut\ D$. Assume that the orders of g and h are equal and odd, and $\rho(g) = h$ for some $\rho \in Aut\ A$. Then*

$$Iso(D,A,g,\Gamma) = Iso(D,A,h,\Gamma).$$

Let $p(>3)$ be prime and $\mathbf{Z}_p$ the cyclic group of order p. Then $\mathbf{Z}_p^3 = \mathbf{Z}_p \times \mathbf{Z}_p \times \mathbf{Z}_p$ has the IEP. Since $\mathbf{Z}_p^3$ is the 3-dimensional vector space over $\mathbf{Z}_p$, the general linear group $GL_3(\mathbf{Z}_p)$ is the automorphism group of $\mathbf{Z}_p^3$. Furthermore, $GL_3(\mathbf{Z}_p)$ acts transitively on $\mathbf{Z}_p^3 \setminus \{\mathbf{0}\}$. Set $\mathbf{e} = \mathbf{e}_1 = (100)^t \in \mathbf{Z}_p^3$. By Theorem 2, we have $Iso(D, A, g, \Gamma) = Iso(D, A, \mathbf{e}, \Gamma)$ for any element $g \in \mathbf{Z}_p^3 \setminus \{\mathbf{0}\}$. Thus we consider the number of Γ-isomorphism classes of $\mathbf{e}$-cyclic $\mathbf{Z}_p^3$-covers of D.

Let $\Gamma \leq Aut\ D$ and $\Pi = GL_3(\mathbf{Z}_p)$. Furthermore, set

$$\Pi_{\mathbf{e}} = \{\sigma \in \Pi \mid \sigma(\mathbf{e}) = \mathbf{e}\}.$$

An action of $\Pi_{\mathbf{e}} \times \Gamma$ on $H^1(G; \mathbf{Z}_p^3)$ is defined as follows:

$$(\mathbf{A}, \gamma)[\alpha] = [\mathbf{A}\alpha^\gamma] = \{\mathbf{A}\alpha^\gamma + \delta s \mid s \in C^0(G; \mathbf{Z}_p^3)\},$$

where $\mathbf{A} \in \Pi_{\mathbf{e}}$, $\gamma \in \Gamma$ and $\alpha \in C^1(G; \mathbf{Z}_p^3)$. By Theorem 1, the number of Γ-isomorphism classes of $\mathbf{e}$-cyclic $\mathbf{Z}_p^3$-covers of D is equal to that of $\Pi_{\mathbf{e}} \times \Gamma$-orbits on $H^1(G; \mathbf{Z}_p^3)$.

Let $\lambda \in \mathbf{Z}_p^*$ and i an integer. Then we introduced two types of matrices as follows:

$$D_{n,\lambda} = \begin{bmatrix} \lambda & 0 & 0 & \cdots & 0 & 0 \\ 1 & \lambda & 0 & \cdots & 0 & 0 \\ 0 & 1 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & \lambda & 0 \\ 0 & 0 & 0 & \cdots & 1 & \lambda \end{bmatrix}, \mathbf{B}_2^i,$$

where $0 \leq i < p^2 - 1, i \not\equiv 0\ (mod\ p+1)$.

Let $K = GF(p^2)$ be the finite field with p^2 elements, and ρ be a generator of the cyclic multiplicative group K^*. Then, identifying $GL_2(\mathbf{Z}_p)$ with $GL(K)$, the matrix $\mathbf{B}_2$ is defined as follows:

$$\mathbf{B}_2(\tau) = \rho\tau, \tau \in K\ (c.f., [11]).$$

Note that $ord(\mathbf{B}_2) = p^2 - 1$, where $ord(\lambda)$ is the order of λ.

Let D be a connected symmetric digraph, G the underlying graph of D, $\Gamma \leq Aut\ D$, $\gamma \in \Gamma$, $\lambda \in \mathbf{Z}_p^*$ and $0 \leq i < p^2 - 1$ such that $i \not\equiv 0\ (mod\ p+1)$.

A $<\gamma>$-orbit σ of length k on $E(G)$ is called diagonal if $\sigma = <\gamma> \{x, \gamma^k(x)\}$ for some $x \in V(G)$. The vertex orbit $<\gamma> x$ and the arc orbit $<\gamma> (x, \gamma^k(x))$ are also called diagonal.

Let $z = \lambda, \mathbf{B}_2^i$ and $m = ord(z)$. Then a diagonal arc orbit of length $2k$ (the corresponding edge orbit of length k and the corresponding vertex orbit of length $2k$) is called type-1 if $z^k = -1$ or $z^k = -\mathbf{I}$, and type-2 otherwise.

Let $k \in \mathbf{N}$. A $<\gamma>$-orbit σ on $V(G), E(G)$ or $D(G)$ is called k-divisible if $|\sigma| \equiv 0\ (mod\ k)$. A $<\gamma>$-orbit σ on $V(G)$ is called edge-induced if there exists a orbit $<\gamma> \{x, y\}$ on $E(G)$ with $x, y \in \sigma$. A k-divisible $<\gamma>$-orbit σ on $V(G)$ is called strongly k-divisible if σ is edge-induced and satisfies the following condition:

If $\Omega = <\gamma> (x, y)$ is any $<\gamma>$-orbit on $D(G)$, and $y = \gamma^j(x)$, $x, y \in \sigma$, then $j \equiv 0\ (mod\ k)$.

Note that, if $\sigma =< \gamma > x$ is strongly m-divisible, $| \sigma |= t$ and there exists a diagonal $< \gamma >$-orbit $\Omega =< \gamma > (x, \gamma^{t/2}(x))$ on $D(G)$, then Ω is type-2.

For $\gamma \in \Gamma$, let $G(\gamma)$ be a simple graph whose vertices are the $< \gamma >$-orbits on $V(G)$, with two vertices adjacent in $G(\gamma)$ if and only if some two of their representatives are adjacent in G. The k th p-level of $G(\gamma)$ is the induced subgraph of $G(\gamma)$ on the vertices ω such that $\theta(| \omega |) = p^k$, where $\theta(i)$ is the largest power of p dividing i. A p-level component of $G(\gamma)$ is a connected component of some p-level of $G(\gamma)$.

Let $z = \lambda, \mathbf{B}_2^i$ and $m = ord(z)$. Then, let $G_z(\gamma)$ be the subgraph induced by the set of m-divisible $< \gamma >$-orbits on $V(G)$. Note that $G_1(\gamma) = G(\gamma)$ for $z = \lambda = 1$. The k th p-level and p-level components of $G_z(\gamma)$ are defined similarly to the case of $G(\gamma)$. A p-level component K of $G_z(\gamma)$ is is called minimal if each σ of K satisfies the condition $\theta(| \sigma |) < \theta(| \omega |)$ whenever $\omega \notin K$ and $\sigma\omega \in E(G(\gamma))$ (c.f., [1]).

Let H be a p-level component of $G_z(\gamma)$ (a 0 th p-level component of $G_\lambda(\gamma)$). Then H is called z-favorable(0-favorable) if H satisfies one of the following conditions:

1. H is not minimal,
2. some σ of H is not strongly m-divisible, or
3. some σ of H is type-1 diagonal.

Otherwise H is called z-defective(0-defective). If $z = \lambda$, then z-defective(z-favorable) p-level components of $G_z(\gamma)$ are defective(favorable) p-level components of $G_\lambda(\gamma)$(see [21]).

Let $\lambda \in \mathbf{Z}_p^*$ and $m = ord(\lambda)$. For $k \geq 1$, let H be a k th p-level component of $G_\lambda(\gamma)$. Then H is called λ-semidefective if H is not λ-favorable, and some σ of H is strongly m-divisible but not strongly p-divisible. Furthermore, H is called λ-strongly defective if H is minimal, and each vertex of H is strongly pm-divisible but not type-1 diagonal.

Theorem 3 *Let D be a connected symmetric digraph, G its underlying graph, $p(> 3)$ prime, $g \in \mathbf{Z}_p^3 \setminus \{\mathbf{0}\}$ and $\Gamma \leq Aut\ D$. Let $z = \lambda, \mathbf{B}_2^i$, where $\lambda \in \mathbf{Z}_p^*$, $0 \leq i < p^2 - 1, i \not\equiv 0 \pmod{p+1}$ and $m = ord(z)$. For $\gamma \in \Gamma$ and z, let $\epsilon(\gamma)$, $\kappa(\gamma, z), \mu(\gamma, z), \nu(\gamma, z)$ and $d(\gamma, z)$ be the number of $< \gamma >$-orbits on $E(G)$, not m-divisible, not diagonal $< \gamma >$-orbits on $E(G)$, type-2 diagonal $< \gamma >$-orbits on $E(G)$, m-divisible $< \gamma >$-orbits on $V(G)$, and z-defective p-level component of $G_z(\gamma)$, respectively. For $\gamma \in \Gamma$ and $\lambda \in \mathbf{Z}_p^*$, let $\nu_0(\gamma, \lambda)$, $\mu_1(\gamma, \lambda)$ and $\kappa_1(\gamma, \lambda)$ be the number of m-divisible, not p-divisible $< \gamma >$-orbits on $V(G)$, p-divisible type-1 diagonal $< \gamma >$-orbits on $E(G)$, and pm-divisible, not diagonal $< \gamma >$-orbits on $E(G)$, respectively. Furthermore, let $c(\gamma, \lambda), d_1(\gamma, \lambda)$, $d_2(\gamma, \lambda)$ and $d_0(\gamma, \lambda)$ be the number of p-level components, λ-favorable p-level components, λ-strongly defective p-level components and 0-favorable p-level components of $G_\lambda(\gamma)$, respectively. Then the number of Γ-isomorphism classes of g-cyclic $\mathbf{Z}_p^3$-covers of D*

is

$$Iso(D,\mathbf{Z}_p^3,g,\Gamma) = \frac{1}{p^3(p-1)^2(p+1)|\Gamma|}\sum_{\gamma\in\Gamma}\{p^{3(\epsilon(\gamma)-\nu(\gamma,1)-\kappa(\gamma,1)-\mu(\gamma,1)+d(\gamma,1))}$$

$$+ (p+1)^2(p-1)p^{2\epsilon(\gamma)-3\nu(\gamma,1)+\nu_0(\gamma,1)-2\mu(\gamma,1)+\kappa_1(\gamma,1)-2\kappa(\gamma,1)+\mu_1(\gamma,1)+c(\gamma,1)-d_1(\gamma,1)+d_2(\gamma,1)-d_0(\gamma,1)+d(\gamma,1)}$$

$$+ p(p-1)^2(p+1)p^{\epsilon(\gamma)-3\nu(\gamma,1)+2\nu_0(\gamma,1)-\mu(\gamma,1)+2\kappa_1(\gamma,1)-\kappa(\gamma,1)+2\mu_1(\gamma,1)+c(\gamma,1)-d_1(\gamma,1)+2d_2(\gamma,1)-d_0(\gamma,1)}$$

$$+ p^2(p+1)\sum_{\lambda=2}^{p-1}p^{3\epsilon(\gamma)-2\nu(\gamma,1)-\nu(\gamma,\lambda)-2\kappa(\gamma,1)-\kappa(\gamma,\lambda)-2\mu(\gamma,1)-\mu(\gamma,\lambda)+2d(\gamma,1)+d(\gamma,\lambda)}$$

$$+ p^2(p-1)(p+1)\sum_{\lambda=2}^{p-1}p^{2\epsilon(\gamma)-2\nu(\gamma,1)+\nu_0(\gamma,1)-\mu(\gamma,1)+\kappa_1(\gamma,1)-\kappa(\gamma,1)+\mu_1(\gamma,1)+c(\gamma,1)-d_1(\gamma,1)+d_2(\gamma,1)-d_0(\gamma,1)-\nu(\gamma,\lambda)-\mu(\gamma,\lambda)-\kappa(\gamma,\lambda)+d(\gamma,\lambda)}$$

$$+ p^2\sum_{\lambda=2}^{p-1}p^{3\epsilon(\gamma)-2\nu(\gamma,\lambda)-\nu(\gamma,1)-2\kappa(\gamma,\lambda)-\kappa(\gamma,1)-2\mu(\gamma,\lambda)-\mu(\gamma,1)+2d(\gamma,\lambda)+d(\gamma,1)}$$

$$+ p^2(p-1)(p+1)\sum_{\lambda=2}^{p-1}p^{2\epsilon(\gamma)-2\nu(\gamma,\lambda)+\nu_0(\gamma,\lambda)-\mu(\gamma,\lambda)+\kappa_0(\gamma,\lambda)-\kappa(\gamma,\lambda)+\mu_1(\gamma,\lambda)+c(\gamma,\lambda)-d_1(\gamma,\lambda)+d_2(\gamma,\lambda)-d_0(\gamma,\lambda)-\nu(\gamma,1)-\mu(\gamma,1)-\kappa(\gamma,1)+d(\gamma,1)}$$

$$+ p^3(p+1)\sum_{1<\lambda<\tau\le p-1}p^{3\epsilon(\gamma)-\nu(\gamma,\lambda)-\nu(\gamma,\tau)-\nu(\gamma,1)-\kappa(\gamma,\lambda)-\kappa(\gamma,\tau)-\kappa(\gamma,1)-\mu(\gamma,\lambda)-\mu(\gamma,\tau)-\mu(\gamma,1)+d(\gamma,\lambda)+d(\gamma,\tau)+d(\gamma,1)}$$

$$+ p^3(p-1)\sum_{0<i<p^2-1,i\not\equiv 0(mod\ p+1)}p^{3\epsilon(\gamma)-2\nu(\gamma,\mathbf{B}_2^i)-2\kappa(\gamma,\mathbf{B}_2^i)-2\mu(\gamma,\mathbf{B}_2^i)+2d(\gamma,\mathbf{B}_2^i)-\nu(\gamma,1)-\kappa(\gamma,1)-\mu(\gamma,1)+d(\gamma,1)}\},$$

where we select only one of i and i' such that $ip \equiv i'(mod\ p^2-1)$ in the last summation.

Proof. Let $\Pi = GL_3(\mathbf{Z}_p)$. By the preceding remark and Burnside's Lemma, the number of Γ-isomorphism classes of $\mathbf{e}$-cyclic $\mathbf{Z}_p^3$-covers of D is

$$\frac{1}{|\Pi_{\mathbf{e}}|\cdot|\Gamma|}\sum_{(\mathbf{A},\gamma)\in\Pi_{\mathbf{e}}\times\Gamma}|H^1(G;\mathbf{Z}_p^3)^{(\mathbf{A},\gamma)}|,$$

where $U^{(\mathbf{A},\gamma)}$ is the set consisting of the elements of U fixed by $(\mathbf{A},\gamma)$.

Now, we have

$$\Pi_{\mathbf{e}} = \{\begin{bmatrix} 1 & a & b \\ \mathbf{0} & & \mathbf{B}\end{bmatrix} \mid a,b=0,1,\cdots,p-1;\mathbf{B}\in GL_2(\mathbf{Z}_p)\}.$$

Now, there exist p^2+p-1 conjugacy classes of $\Pi_{\mathbf{e}}$. By Theorem 4, the representatives of these conjugacy classes are given as follows:

$$\mathbf{A}_1=\mathbf{I}_3=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\mathbf{A}_2=\begin{bmatrix}{}^t\mathbf{D}_{2,1}&\\&1\end{bmatrix},\mathbf{A}_3=\begin{bmatrix}1&\\&\mathbf{D}_{2,1}\end{bmatrix},\mathbf{A}_4=\begin{bmatrix}1&0&1\\0&1&0\\0&1&1\end{bmatrix},$$

$$\mathbf{A}_{5,\lambda}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&\lambda\end{bmatrix}(2\leq\lambda\leq p-1),\mathbf{A}_{6,\lambda}=\begin{bmatrix}{}^t\mathbf{D}_{2,1}&\\&\lambda\end{bmatrix}(2\leq\lambda\leq p-1),$$

$$\mathbf{A}_{7,\lambda}=\begin{bmatrix}1&0&0\\0&\lambda&0\\0&0&\lambda\end{bmatrix}(2\leq\lambda\leq p-1),\mathbf{A}_{8,\lambda}=\begin{bmatrix}1&\\&\mathbf{D}_{2,\lambda}\end{bmatrix}(2\leq\lambda\leq p-1),$$

$$\mathbf{A}_{9,\lambda,\tau}=\begin{bmatrix}1&0&0\\0&\lambda&0\\0&0&\tau\end{bmatrix}(2\leq\lambda\neq\tau\leq p-1),\mathbf{A}_{10,i}=\begin{bmatrix}1&\\&\mathbf{B}_2^i\end{bmatrix}(0<i<p^2-1,i\not\equiv 0\,(mod\,p+1)).$$

where we select only one of i and i' such that $ip\equiv i'(mod\,p^2-1)$ for $\mathbf{B}_2^i$ (see [11]). The cardinalities of the first, second,$\cdots$, tenth type of conjugacy classes are as follows:

$$1,p^2-1,p(p-1)(p+1),p(p-1)^2(p+1),p^2(p+1),p^2(p-1)(p+1),$$
$$p^2,p^2(p-1)(p+1),p^3(p+1),p^3(p-1).$$

Furthermore, the number of the first, second,$\cdots$, tenth type of conjugacy classes are as follows:

$$1,1,1,1,p-2,p-2,p-2,p-2,\frac{1}{2}(p-2)(p-3),\frac{1}{2}p(p-1).$$

The detail is developed in Section 3.

Let $\mathbf{A},\mathbf{F}\in\Pi_{\mathbf{e}}$ be conjugate. Then there exists an element $\mathbf{C}\in\Pi_{\mathbf{e}}$ such that $\mathbf{CAC}^{-1}=\mathbf{F}$. Thus $[\alpha]\in H^1(G;\mathbf{Z}_p^3)^{(\mathbf{A},\gamma)}$ if and only if $\mathbf{A}\alpha^\gamma=\alpha+\delta s$ for some $s\in C^0(G;\mathbf{Z}_p^3)$. But $\mathbf{A}\alpha^\gamma=\alpha+\delta s$ if and only if $\mathbf{F}(\mathbf{C}\alpha)^\gamma=\mathbf{C}\alpha+\delta(\mathbf{C}s)$, i.e., $[\mathbf{C}\alpha]\in H^1(G;\mathbf{Z}_p^3)^{(\mathbf{F},\gamma)}$. By the fact that a mapping $[\alpha]\longmapsto[\mathbf{C}\alpha]$ is bijective, we have

$$|\,H^1(G;\mathbf{Z}_p^3)^{(\mathbf{A},\gamma)}\,|=|\,H^1(G;\mathbf{Z}_p^3)^{(\mathbf{F},\gamma)}\,|\,.$$

Therefore the number of Γ-isomorphism classes of $\mathbf{e}$-cyclic $\mathbf{Z}_p^3$-covers of D is

$$Iso(D,\mathbf{Z}_p^3,\mathbf{e},\Gamma)=\frac{1}{p^3(p-1)^2(p+1)\,|\,\Gamma\,|}\sum_{\gamma\in\Gamma}\{|\,H^1(G;\mathbf{Z}_p^3)^{(\mathbf{A}_1,\gamma)}\,|$$

$$+(p-1)(p+1)\,|\,H^1(G;\mathbf{Z}_p^3)^{(\mathbf{A}_2,\gamma)}\,|+p(p-1)(p+1)\,|\,H^1(G;\mathbf{Z}_p^3)^{(\mathbf{A}_3,\gamma)}\,|+p(p-1)^2(p+1)\times$$

$$|\,H^1(G;\mathbf{Z}_p^3)^{(\mathbf{A}_4,\gamma)}\,|+p^2(p+1)\sum_{\lambda=2}^{p-1}|\,H^1(G;\mathbf{Z}_p^3)^{(\mathbf{A}_{5,\lambda},\gamma)}\,|+p^2(p-1)(p+1)\sum_{\lambda=2}^{p-1}|\,H^1(G;\mathbf{Z}_p^3)^{(\mathbf{A}_{6,\lambda},\gamma)}\,|$$

$$+p^2\sum_{\lambda=2}^{p-1} | H^1(G;\mathbf{Z}_p^3)^{(\mathbf{A}_{7,\lambda},\gamma)} | +p^2(p-1)(p+1)\sum_{\lambda=2}^{p-1} | H^1(G;\mathbf{Z}_p^3)^{(\mathbf{A}_{8,\lambda},\gamma)} |$$

$$+p^3(p+1)\sum_{2\le\lambda<\tau\le p-1} | H^1(G;\mathbf{Z}_p^3)^{(\mathbf{A}_{9,\lambda,\tau},\gamma)} | +p^3(p-1)\sum_{i} | H^1(G;\mathbf{Z}_p^3)^{(\mathbf{A}_{10,i},\gamma)} |\}.$$

Let $(\mathbf{A},\gamma)\in\Pi_{\mathbf{e}}\times\Gamma$.

Case 1: $\mathbf{A}=\mathbf{A}_1=\mathbf{I}_3$.

Then $[\alpha]\in H^1(G;\mathbf{Z}_p^3)^{(\mathbf{A}_1,\gamma)}$ if and only if $\mathbf{A}_1\alpha^\gamma=\alpha+\delta s$ for some $s\in C^0(G;\mathbf{Z}_p^3)$.

Now, let $\alpha=a\mathbf{e}_1+b\mathbf{e}_2+c\mathbf{e}_3$, $a,b,c\in C^1(G;\mathbf{Z}_p)$, where $\mathbf{e}_1={}^t(100),\mathbf{e}_2={}^t(010)$ and $\mathbf{e}_3={}^t(001)$. Furthermore, let $s=v\mathbf{e}_1+w\mathbf{e}_2+z\mathbf{e}_3$, $v,w,z\in C^0(G;\mathbf{Z}_p)$. Then $\alpha^\gamma=\alpha+\delta s$ if and only if $a^\gamma=a+\delta v$, $b^\gamma=b+\delta w$, and $c^\gamma=c+\delta z$, i.e., $(1,\gamma)[a]=[a]$, $(1,\gamma)[b]=[b]$ and $(1,\gamma)[c]=[c]$. Note that $[a],[b],[c]\in H^1(G;\mathbf{Z}_p)^{(1,\gamma)}$. Since $[a\mathbf{e}_1+b\mathbf{e}_2+c\mathbf{e}_3]=[a]\mathbf{e}_1+[b]\mathbf{e}_2+[c]\mathbf{e}_3$, we have

$$| H^1(G;\mathbf{Z}_p^3)^{(\mathbf{A}_1,\gamma)} |=| H^1(G;\mathbf{Z}_p)^{(1,\gamma)} |^3.$$

By Theorem 3.3 of [21], it follows that

$$| H^1(G;\mathbf{Z}_p^3)^{(\mathbf{A}_1,\gamma)} |= p^{3(\epsilon(\gamma)-\nu(\gamma,1)-\kappa(\gamma,1)-\mu(\gamma,1)+d(\gamma,1))}.$$

Case 2: $\mathbf{A}=\mathbf{A}_2$.

Similarly to case 1, we have

$$| H^1(G;\mathbf{Z}_p^3)^{(\mathbf{A}_2,\gamma)} |=| H^1(G;\mathbf{Z}_p^2)^{({}^t\mathbf{D}_{2,1},\gamma)} |\cdot| H^1(G;\mathbf{Z}_p)^{(1,\gamma)} |.$$

But, ${}^t\mathbf{D}_{2,1}$ and $\mathbf{D}_{2,1}$ are conjugate in $GL_2(\mathbf{Z}_p)$, and so

$$| H^1(G;\mathbf{Z}_p^2)^{({}^t\mathbf{D}_{2,1},\gamma)} |=| H^1(G;\mathbf{Z}_p^2)^{(\mathbf{D}_{2,1},\gamma)} |.$$

By Theorem 3 of [20] and Theorem 3.3 of [21], it follows that

$$| H^1(G;\mathbf{Z}_p^3)^{(\mathbf{A}_2,\gamma)} |= p^{2\epsilon(\gamma)-3\nu(\gamma,1)+\nu_0(\gamma,1)-2\mu(\gamma,1)+\kappa_1(\gamma,1)-2\kappa(\gamma,1)+\mu_1(\gamma,1)+c(\gamma,1)-d_1(\gamma,1)+d_2(\gamma,1)-d_0(\gamma,1)+d(\gamma,1)}$$

Case 3: $\mathbf{A}=\mathbf{A}_3$.

Then we have

$$| H^1(G;\mathbf{Z}_p^3)^{(\mathbf{A}_3,\gamma)} |=| H^1(G;\mathbf{Z}_p)^{(1,\gamma)} |\cdot| H^1(G;\mathbf{Z}_p^2)^{(\mathbf{D}_{2,1},\gamma)} |.$$

By case 2, it follows that

$$| H^1(G;\mathbf{Z}_p^3)^{(\mathbf{A}_3,\gamma)} |=| H^1(G;\mathbf{Z}_p^3)^{(\mathbf{A}_2,\gamma)} |.$$

Case 4: $\mathbf{A}=\mathbf{A}_4$.

Then $\mathbf{D}_{3,1}$ and $\mathbf{A}_4$ are conjugate in $GL_3(\mathbf{Z}_p)_{\mathbf{e}}$, and so

$$| H^1(G;\mathbf{Z}_p^3)^{(\mathbf{A}_4,\gamma)} |=| H^1(G;\mathbf{Z}_p^3)^{(\mathbf{D}_{3,1},\gamma)} |.$$

By Theorem 3 of [20], it follows that

$$| H^1(G;\mathbf{Z}_p^3)^{(\mathbf{A}_4,\gamma)} |= p^{\epsilon(\gamma)-3\nu(\gamma,1)+2\nu_0(\gamma,1)-\mu(\gamma,1)+2\kappa_1(\gamma,1)-\kappa(\gamma,1)+2\mu_1(\gamma,1)+c(\gamma,1)-d_1(\gamma,1)+2d_2(\gamma,1)-d_0(\gamma,1)}.$$

Case 5: $\mathbf{A}=\mathbf{A}_{5,\lambda}$.
Then we have

$$| H^1(G;\mathbf{Z}_p^3)^{(\mathbf{A}_{5,\lambda},\gamma)} |=| H^1(G;\mathbf{Z}_p)^{(1,\gamma)} |^2 \cdot | H^1(G;\mathbf{Z}_p)^{(\lambda,\gamma)} |.$$

By Theorem 3.3 of [21], it follows that

$$| H^1(G;\mathbf{Z}_p^3)^{(\mathbf{A}_{5,\lambda},\gamma)} |= p^{3\epsilon(\gamma)-2\nu(\gamma,1)-\nu(\gamma,\lambda)-2\kappa(\gamma,1)-\kappa(\gamma,\lambda)-2\mu(\gamma,1)-\mu(\gamma,\lambda)+2d(\gamma,1)+d(\gamma,\lambda)}.$$

Case 6: $\mathbf{A}=\mathbf{A}_{6,\lambda}$.
Then we have

$$| H^1(G;\mathbf{Z}_p^3)^{(\mathbf{A}_{6,\lambda},\gamma)} |=| H^1(G;\mathbf{Z}_p^2)^{(\mathbf{D}_{2,1},\gamma)} | \cdot | H^1(G;\mathbf{Z}_p)^{(\lambda,\gamma)} |.$$

By Theorem 3 of [20] and Theorem 3.3 of [21], it follows that

$$| H^1(G;\mathbf{Z}_p^3)^{(\mathbf{A}_{6,\lambda,\tau},\gamma)} |= p^{2\epsilon(\gamma)-2\nu(\gamma,1)+\nu_0(\gamma,1)-\mu(\gamma,1)+\kappa_1(\gamma,1)-\kappa(\gamma,1)+\mu_1(\gamma,1)+c(\gamma,1)-d_1(\gamma,1)+d_2(\gamma,1)-d_0(\gamma,1)-\nu(\gamma,\lambda)-\mu(\gamma,\lambda)-\kappa(\gamma,\lambda)+d(\gamma,\lambda)}.$$

Case 7: $\mathbf{A}=\mathbf{A}_{7,\lambda}$.
Then we have

$$| H^1(G;\mathbf{Z}_p^3)^{(\mathbf{A}_{7,\lambda},\gamma)} |=| H^1(G;\mathbf{Z}_p)^{(1,\gamma)} | \cdot | H^1(G;\mathbf{Z}_p)^{(\lambda,\gamma)} |^2.$$

By Theorem 3.3 of [21], it follows that

$$| H^1(G;\mathbf{Z}_p^3)^{(\mathbf{A}_{7,\lambda},\gamma)} |= p^{3\epsilon(\gamma)-2\nu(\gamma,\lambda)-\nu(\gamma,1)-2\kappa(\gamma,\lambda)-\kappa(\gamma,1)-2\mu(\gamma,\lambda)-\mu(\gamma,1)+2d(\gamma,\lambda)+d(\gamma,1)}.$$

Case 8: $\mathbf{A}=\mathbf{A}_{8,\lambda}$.
Then we have

$$| H^1(G;\mathbf{Z}_p^3)^{(\mathbf{A}_{8,\lambda},\gamma)} |=| H^1(G;\mathbf{Z}_p^2)^{(\mathbf{D}_{2,\lambda},\gamma)} | \cdot | H^1(G;\mathbf{Z}_p)^{(1,\gamma)} |.$$

By Theorem 3 of [20] and Theorem 3.3 of [21], it follows that

$$| H^1(G;\mathbf{Z}_p^3)^{(\mathbf{A}_{8,\lambda},\gamma)} |= p^{2\epsilon(\gamma)-2\nu(\gamma,\lambda)+\nu_0(\gamma,\lambda)-\mu(\gamma,\lambda)+\kappa_1(\gamma,\lambda)-\kappa(\gamma,\lambda)+\mu_1(\gamma,\lambda)+c(\gamma,\lambda)-d_1(\gamma,\lambda)+d_2(\gamma,\lambda)-d_0(\gamma,\lambda)-\nu(\gamma,1)-\mu(\gamma,1)-\kappa(\gamma,1)+d(\gamma,1)}.$$

Case 9: $\mathbf{A}=\mathbf{A}_{9,\lambda,\tau}$.
Then we have

$$| H^1(G;\mathbf{Z}_p^3)^{(\mathbf{A}_{9,\lambda,\tau},\gamma)} |=| H^1(G;\mathbf{Z}_p)^{(1,\gamma)} | \cdot | H^1(G;\mathbf{Z}_p)^{(\lambda,\gamma)} | \cdot | H^1(G;\mathbf{Z}_p)^{(\tau,\gamma)} |.$$

By Theorem 3.3 of [21], it follows that

$$| H^1(G;\mathbf{Z}_p^3)^{(\mathbf{A}_{9,\lambda,\tau},\gamma)} |= p^{3\epsilon(\gamma)-\nu(\gamma,1)-\nu(\gamma,\lambda)-\nu(\gamma,\tau)-\kappa(\gamma,1)-\kappa(\gamma,\lambda)-\kappa(\gamma,\tau)}$$
$$^{-\mu(\gamma,1)-\mu(\gamma,\lambda)-\mu(\gamma,\tau)+d(\gamma,1)+d(\gamma,\lambda)+d(\gamma,\tau)}.$$

Case 10: $\mathbf{A} = \mathbf{A}_{10,i}$.
Then we have

$$| H^1(G;\mathbf{Z}_p^3)^{(\mathbf{A}_{10,i},\gamma)} |=| H^1(G;\mathbf{Z}_p)^{(1,\gamma)} | \cdot | H^1(G;\mathbf{Z}_p^2)^{(\mathbf{B}_2^i,\gamma)} | .$$

By Theorem 4 of [20] and Theorem 3.3 of [21], it follows that

$$| H^1(G;\mathbf{Z}_p^3)^{(\mathbf{A}_{10,i},\gamma)} |= p^{3\epsilon(\gamma)-2\nu(\gamma,\mathbf{B}_2^i)-2\kappa(\gamma,\mathbf{B}_2^i)-2\mu(\gamma,\mathbf{B}_2^i)+2d(\gamma,\mathbf{B}_2^i)}$$
$$^{-\nu(\gamma,1)-\mu(\gamma,1)-\kappa(\gamma,1)+d(\gamma,1)}.$$

By cases 1,2, $\cdots$, 9 and 10, the result follows. Q.E.D.

Corollary 1 *Let D be a connected symmetric digraph, G its underlying graph, $p(>3)$ prime and $g \in \mathbf{Z}_p^3$. Then the number of I-isomorphism classes of g-cyclic $\mathbf{Z}_p^3$-covers of D is*

$$\begin{aligned} Iso(D,\mathbf{Z}_p^3,g,I) &= \frac{1}{p^3(p-1)^2(p+1)}\{p^{3B} + (p+1)(p^3-p^2-1)p^{2B} \\ &+ p(p^5-p^4-2p^3+p^2+p+1)p^B\}, \end{aligned}$$

where $B = B(G) =| E(G) | - | V(G) | +1$ is the Betti-number of G.

Proof. Since $I = \{1\}$, we have $\epsilon(1) =| E(G) |$, $\mu(1,z) = \mu_1(1,\lambda) = \kappa_1(1,\lambda) = d_1(1,\lambda) = d_0(1,\lambda) = d_2(1,\lambda) = 0$, where $z = \lambda, \mathbf{B}_2^i$. Moreover, we have

$$\nu(1,z) = \begin{cases} | V(G) | & \text{if } z=\lambda=1, \\ 0 & \text{otherwise,} \end{cases} \qquad \kappa(1,z) = \begin{cases} 0 & \text{if } z=\lambda=1\ , \\ | E(G) | & \text{otherwise,} \end{cases}$$

$$\nu_0(1,\lambda) = \begin{cases} | V(G) | & \text{if } \lambda=1, \\ 0 & \text{otherwise} \end{cases} \qquad and \ \ c(1,\lambda) = \begin{cases} 1 & \text{if } \lambda=1\ , \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, we have

$$d(1,z) = \begin{cases} 1 & \text{if } z=\lambda=1\ , \\ 0 & \text{otherwise.} \end{cases}$$

Q.E.D.

This is the formula for $r = 3$ in [16, Theorem 4.4].

3. The conjugacy classes of $GL_3(\mathbf{Z}_p)_{\mathbf{e}}$

Let p an odd prime number. Then we consider all conjugacy classes of $GL_3(\mathbf{Z}_p)_{\mathbf{e}}$, where $\mathbf{e} = (100)^t$.

Theorem 4 *Let p be an odd prime. Then the representatives of the conjugacy classes of $GL_3(\mathbf{Z}_p)_{\mathbf{e}}$ are given as follows:*

$$\mathbf{A}_1 = \mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathbf{A}_2 = \begin{bmatrix} {}^t\mathbf{D}_{2,1} & \\ & 1 \end{bmatrix}, \mathbf{A}_3 = \begin{bmatrix} 1 & \\ & \mathbf{D}_{2,1} \end{bmatrix}, \mathbf{A}_4 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

$$\mathbf{A}_{5,\lambda} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{bmatrix} (2 \le \lambda \le p-1), \mathbf{A}_{6,\lambda} = \begin{bmatrix} {}^t\mathbf{D}_{2,1} & \\ & \lambda \end{bmatrix} (2 \le \lambda \le p-1),$$

$$\mathbf{A}_{7,\lambda} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} (2 \le \lambda \le p-1), \mathbf{A}_{8,\lambda} = \begin{bmatrix} 1 & \\ & \mathbf{D}_{2,\lambda} \end{bmatrix} (2 \le \lambda \le p-1),$$

$$\mathbf{A}_{9,\lambda,\tau} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \tau \end{bmatrix} (2 \le \lambda \ne \tau \le p-1), \mathbf{A}_{10,i} = \begin{bmatrix} 1 & \\ & \mathbf{B}_2^i \end{bmatrix} (0 < i < p^2-1, i \not\equiv 0 \ (mod\, p+1)).$$

Proof. Let $\Pi = GL_3(\mathbf{Z}_p)$. Then we have

$$\Pi_{\mathbf{e}} = \{\begin{bmatrix} 1 & a \ b \\ \mathbf{0} & \mathbf{B} \end{bmatrix} \mid a, b = 0, 1, \cdots, p-1; \mathbf{B} \in GL_2(\mathbf{Z}_p)\}.$$

Now, let

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \ 0 \\ \mathbf{0} & \mathbf{B} \end{bmatrix}, \mathbf{F} = \begin{bmatrix} 1 & c \ d \\ \mathbf{0} & \mathbf{D} \end{bmatrix}, \mathbf{B} = \begin{bmatrix} s & t \\ u & v \end{bmatrix}, \mathbf{D} = \begin{bmatrix} k & l \\ m & n \end{bmatrix},$$

where $c, d = 0, 1, \cdots, p-1; \mathbf{B}, \mathbf{D} \in GL_2(\mathbf{Z}_p)$. Then we have

$$\mathbf{F}^{-1}\mathbf{A}\mathbf{F} = \begin{bmatrix} 1 & \alpha \ \beta \\ \mathbf{0} & \mathbf{D}^{-1}\mathbf{B}\mathbf{D} \end{bmatrix},$$

where

$$\alpha = c + 1/\mid \mathbf{D} \mid \{-(cn-dm)(sk+tm) + (cl-dk)(uk+vm)\},$$

$$\beta = d + 1/\mid \mathbf{D} \mid \{-(cn-dm)(sl+tn) + (cl-dk)(ul+vn)\}.$$

Next, we consider the condition for $\mathbf{B}$ and $\mathbf{D}$ to satisfy the equation $\mathbf{D}^{-1}\mathbf{B}\mathbf{D} = \mathbf{B}$. Suppose that $\mathbf{B}\mathbf{D} = \mathbf{D}\mathbf{B}$. Then we have

$$\mathbf{B}\mathbf{D} = \begin{bmatrix} sk+tm & sl+tn \\ uk+vm & ul+vn \end{bmatrix}, \mathbf{D}\mathbf{B} = \begin{bmatrix} sk+ul & tk+vl \\ sm+un & tm+vn \end{bmatrix}.$$

If $\mathbf{BD}=\mathbf{DB}$, then

$$\begin{cases} sk+tm=sk+ul \\ sl+tn=tk+vl \\ uk+vm=sm+un \\ ul+vn=tm+vn \end{cases} \quad i.e., \begin{cases} ul=tm \\ t(n-k)=l(v-s) \\ u(n-k)=m(v-s) \end{cases}.$$

Thus, the $(1,2)$-array of $\mathbf{F}^{-1}\mathbf{AF}$ is

$$c+1/\mid \mathbf{D}\mid (lm-nk)(cs+du)=(1-s)c-ud.$$

Furthermore, the $(1,3)$-array of $\mathbf{F}^{-1}\mathbf{AF}$ is

$$d+1/\mid \mathbf{D}\mid (lm-nk)(ct+dv)=-tc+(1-v)d.$$

For any $a,b\in \mathbf{Z}_p$, set

$$\begin{cases} (1-s)c-ud=a \\ -tc+(1-v)d=b \end{cases} \qquad \cdots (*).$$

Then, there exist $c,d\in \mathbf{Z}_p$ satisfying $(*)$ if and only if

$$\det(\mathbf{I}-\mathbf{B})=\det(\mathbf{I}-\mathbf{B}^t)=\det\begin{bmatrix} 1-s & -u \\ -t & 1-v \end{bmatrix}\neq 0,$$

i.e., $\mathbf{I}-\mathbf{B}$ is regular. Note that $\mathbf{I}-\mathbf{B}$ is not regular if and only if 1 is one of the eigenvalues of $\mathbf{B}$.

But, the representatives of conjugacy classes of $GL_2(\mathbf{Z}_p)$ are given as follows:

$$\mathbf{C}_\lambda=\begin{bmatrix} \lambda & \\ & \lambda \end{bmatrix}(1\leq\lambda\leq p-1), \mathbf{D}_{2,\lambda}=\begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}(1\leq\lambda\leq p-1),$$

$$\mathbf{B}_{\lambda,\tau}=\begin{bmatrix} \lambda & \\ & \tau \end{bmatrix}(1\leq\lambda\neq\tau\leq p-1), \mathbf{B}_2^i(0<i<p^2-1, i\not\equiv 0\ (mod\ p+1)),$$

where we select only one of i and i' such that $ip\equiv i'(mod\ p^2-1)$ for $\mathbf{B}_2^i$ (see [11]). The cardinalities and the number of the first, second,$\cdots$, fourth type of conjugacy classes are as follows:

$$1,(p-1)(p+1),p(p+1),p(p-1);p-1,p-1,\frac{1}{2}(p-1)(p-2),\frac{1}{2}p(p-1).$$

Then each of matrices $\mathbf{C}_\lambda$, $\mathbf{D}_{2,\lambda}(2\leq\lambda\leq p-1)$ and $\mathbf{B}_{\lambda,\tau}(2\leq\lambda\neq\tau\leq p-1)$ does not contain 1 as its eigenvalue. By Lemma 5 of [20], each $\mathbf{B}_2^i(0<i<p^2-1, i\not\equiv 0\ (mod\ p+1))$ does not contain 1 as its eigenvalue. Thus,

$$\mathbf{F}^{-1}\mathbf{AF}=\begin{bmatrix} 1 & a\ b \\ \mathbf{0} & \mathbf{B} \end{bmatrix} \ for\ any\ a,b\in\mathbf{Z}_p,$$

where $\mathbf{B}=\mathbf{C}_\lambda, \mathbf{D}_{2,\lambda}, \mathbf{B}_{\lambda,\tau}, \mathbf{B}_2^i$. Therefore the following four matrices are given as the representatives of conjugacy classes of $GL_2(\mathbf{Z}_p)_\mathbf{e}$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}(2\leq\lambda\leq p-1), \begin{bmatrix} 1 & \\ & \mathbf{D}_{2,\lambda} \end{bmatrix}(2\leq\lambda\leq p-1),$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \tau \end{bmatrix} (2 \le \lambda \ne \tau \le p-1), \begin{bmatrix} 1 & \\ & \mathbf{B}_2^i \end{bmatrix} (0 < i < p^2 - 1, i \not\equiv 0 \ (mod\, p+1)).$$

The cardinalities and the number of the above four types of conjugacy classes are as follows:

$$p^2, p^2(p-1)(p+1), p^3(p+1), p^3(p-1); p-2, p-2, \frac{1}{2}(p-2)(p-3), \frac{1}{2}p(p-1).$$

The representatives of conjugacy classes of $GL_2(\mathbf{Z}_p)$ which contain 1 as eigenvalues are given as follows:

$$\mathbf{C}_1 = \mathbf{I}_2 = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}, \mathbf{D}_{2,1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \mathbf{B}_{1,\lambda} = \begin{bmatrix} 1 & \\ & \lambda \end{bmatrix} (2 \le \lambda \le p-1).$$

Case 1: $\mathbf{B} = \mathbf{C}_1$.

$$\mathbf{F}^{-1}\mathbf{I}_3\mathbf{F} = \mathbf{I}_3 \ for \ any \ \mathbf{F} \in GL_2(\mathbf{Z}_p)_{\mathbf{e}}.$$

Next, let

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathbf{F} = \begin{bmatrix} 1 & c \ d \\ \mathbf{0} & \mathbf{D} \end{bmatrix}, \mathbf{D} = \begin{bmatrix} a & b \\ m & n \end{bmatrix} (an - bm \ne 0).$$

Then we have

$$\mathbf{F}^{-1}\mathbf{A}\mathbf{F} = \begin{bmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

for any $a, b \in \mathbf{Z}_p$.

Therefore the following two matrices are given as the representatives of conjugacy classes of $GL_2(\mathbf{Z}_p)_{\mathbf{e}}$:

$$\mathbf{I}_3, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The cardinalities and the number of the above two types of conjugacy classes are as follows:

$$1, p^2 - 1; 1, 1.$$

Case 2: $\mathbf{B} = \mathbf{D}_{2,1}$.

Then, let

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \mathbf{F} = \begin{bmatrix} 1 & c \ -a \\ \mathbf{0} & \mathbf{D} \end{bmatrix}, \mathbf{D} = \begin{bmatrix} k & 0 \\ m & k \end{bmatrix} (k \ne 0).$$

Then we have

$$\mathbf{F}^{-1}\mathbf{A}\mathbf{F} = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

for any $a \in \mathbf{Z}_p$.

Next, let

$$\mathbf{A}=\begin{bmatrix}1&0&1\\0&1&0\\0&1&1\end{bmatrix},\mathbf{F}=\begin{bmatrix}1&c&-a+m\\\mathbf{0}&&\mathbf{D}\end{bmatrix},\mathbf{D}=\begin{bmatrix}b&0\\m&b\end{bmatrix}(b\neq 0).$$

Then we have

$$\mathbf{F}^{-1}\mathbf{AF}=\begin{bmatrix}1&a&b\\0&1&0\\0&1&1\end{bmatrix}$$

for any $a \in \mathbf{Z}_p$ and $b \in \mathbf{Z}_p^*$.

Therefore the following two matrices are given as the representatives of conjugacy classes of $GL_2(\mathbf{Z}_p)_\mathbf{e}$:

$$\begin{bmatrix}1&0&0\\0&1&0\\0&1&1\end{bmatrix},\begin{bmatrix}1&0&1\\0&1&0\\0&1&1\end{bmatrix}.$$

The cardinalities and the number of the above two types of conjugacy classes are as follows:

$$p(p-1)(p+1),p(p-1)^2(p+1);1,1.$$

Case 3: $\mathbf{B}=\mathbf{B}_{1,\lambda}$.

Then, let

$$\mathbf{A}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&\lambda\end{bmatrix},\mathbf{F}=\begin{bmatrix}1&c&(1-\lambda)^{-1}b\\\mathbf{0}&&\mathbf{D}\end{bmatrix},\mathbf{D}=\begin{bmatrix}k&0\\0&n\end{bmatrix}(kn\neq 0).$$

Then we have

$$\mathbf{F}^{-1}\mathbf{AF}=\begin{bmatrix}1&0&b\\0&1&0\\0&0&\lambda\end{bmatrix}$$

for any $b \in \mathbf{Z}_p$.

Next, let

$$\mathbf{A}=\begin{bmatrix}1&1&0\\0&1&0\\0&0&\lambda\end{bmatrix},\mathbf{F}=\begin{bmatrix}1&c&b(1-\lambda)^{-1}\\\mathbf{0}&&\mathbf{D}\end{bmatrix},\mathbf{D}=\begin{bmatrix}a&0\\0&n\end{bmatrix}(an\neq 0).$$

Then we have

$$\mathbf{F}^{-1}\mathbf{AF}=\begin{bmatrix}1&a&b\\0&1&0\\0&0&\lambda\end{bmatrix}$$

for any $a \in \mathbf{Z}_p^*$ and $b \in \mathbf{Z}_p$.

Therefore the following two matrices are given as the representatives of conjugacy classes of $GL_2(\mathbf{Z}_p)_{\mathbf{e}}$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{bmatrix} (2 \leq \lambda \leq p-1), \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{bmatrix} (2 \leq \lambda \leq p-1).$$

The cardinalities and the number of the above two types of conjugacy classes are as follows:

$$p^2(p+1), p^2(p-1)(p+1); p-2, p-2.$$

Q.E.D.

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