Oscillation of the First Order Differential Equations with Deviating Argument

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進み変数をもつ1階微分方程式のOscillation について

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§1. Introduction

In this note we consider the first order differential equation with deviating argument

$$y'(t)+ay(t)-py(t+\tau)=0 \tag{1}$$

where a, p and τ are constants, $p>0$ and $\tau>0$.

As usual, a solution of equation is called oscillatory if it has arbitrarily large zeros and nonoscillatory if it is eventually positive or eventually negative. Equation is called oscillatory if all of its solutions are oscillatory.

Onose [1] proved some interesting results concerning the oscillations of the first order advanced differential inequalities.

We show that (1) has oscillatory solution under certain condition with similary method.

§2. Mein theorem

Theorem. *Consider the advanced differential equation*

$$y'(t)+ay(t)-py(t+\tau)=0 \tag{1}$$

where a, p and τ are constants, $p>0$ and $\tau>0$.
Assume that

$$p\tau e>e^{a\tau} \tag{2}$$

Then (1) has oscillatory solutions only.

Proof. Let y(t) be a nonoscillatory solution of (1) and assume that for sufficiently large c,

$$y(t)>0,\quad t\geqq c$$

From (1), we obtain

$$e^{at}(y'(t)+ay(t)-py(t+\tau))=0,\quad t\geqq c$$

From this

$$(e^{at}y(t))'-pe^{at}y(t+\tau)=0,\quad t\geqq c$$

If we put

$$X(t)\equiv e^{at}y(t)$$

then equation (1) becomes

(3) $$X'(t)-pe^{-a\tau}X(t+\tau)=0,\quad t\geqq c$$

Since $X(t+\tau)>0$ and $pe^{-a\tau}>0$ for $t\geqq c$, from (3)

(4) $$X'(t)>0,\quad t\geqq c$$

From (4), we obtain

$$X(t)<X(t+\tau),\quad t\geqq c$$

Set

$$w(t)=\frac{X(t+\tau)}{X(t)}(>1),\quad t\geqq c$$

Then,

$$\lambda=\liminf_{t\to\infty} w(t)\geqq 1$$

Dividing (3) by X(t), we obtain

$$\frac{X'(t)}{X(t)}-pe^{-a\tau}\frac{X(t+\tau)}{X(t)}=0$$

from which we have

$$\log X(t+\tau)-\log X(t)=pe^{-a\tau}\int_t^{t+\tau} w(s)\,ds$$

this yields

(5) $$\log w(t)=pe^{-a\tau}\int_t^{t+\tau} w(s)\,ds \qquad \text{for } t\geqq c$$

Now we consider the following two cases:

Case 1: λ is finite.

From (5) we have

(6) $$\log\lambda=p\tau\lambda e^{-a\tau}$$

Using the fact that

$$\max_{\lambda \geqq 1} \frac{\log\lambda}{\lambda} = \frac{1}{e}$$

and (6) we have

$$\frac{\log\lambda}{\lambda} = p\tau e^{-a\tau} \leqq \frac{1}{e}$$

from which it follows that

$$p\tau e \leqq e^{a\tau}$$

this contradicts hypothesis (2).

Case 2: λ is infinite.

That is

(7) $$\lim_{t\to\infty} \frac{X(t+\tau)}{X(t)} = +\infty$$

From (2), we obtain

(8) $$pe^{-a\tau} > \frac{1}{\tau e}$$

Integrating (3) from t to t^*, the next, $t^* - \tau$ to t, $(t^* - \tau < t < t^*)$, and using the fact that X(t) is increasing and (8), we obtain

$$X(t^*) - X(t) = pe^{-a\tau} \int_t^{t^*} X(s+\tau)\,ds > \frac{1}{\tau e} X(t+\tau)$$

and

$$X(t) - X(t^* - \tau) = pe^{-a\tau} \int_{t^*-\tau}^{t} X(s+\tau)\,ds > \frac{1}{\tau e} X(t^*)$$

From two inequalitys, we have

$$X(t) > \frac{1}{\tau e} X(t^*) > \frac{1}{(\tau e)^2} X(t+\tau)$$

From this we obtain

$$\frac{X(t+\tau)}{X(t)} < (\tau e)^2$$

which contradicts to (7).

This completes the proof.

References

[1] Onose, H., Oscillatory Properties of the First Order Differential Inequalities with Deviating Argument, Funkcialaj Ekvacioj, 26(1983), 189-195.

[2] Ladas, G. and Stavroulakis, I.P., On delay differential inequalities of first order, Funkcialaj Ekvaccioj, 25(1982), 105-113.